STATIONARY SETS AND DETERMINING SETS FOR CERTAIN CLASSES OF DARBOUX FUNCTIONS

A. M. BRUCKNER[1] AND JOHN L. LEONARD[2]

1. **Introduction.** Let $\mathfrak{C}$ be a class of real valued functions defined on an interval $[a, b]$. A set E is called *stationary* for $\mathfrak{C}$ provided any member of $\mathfrak{C}$ which is constant on E must be constant on all of $[a, b]$. Several recent articles have been concerned with the problem of determining the stationary sets for various classes of functions [1], [2], [5], [6], [7].

Let us denote by $S(\mathfrak{C})$ the collection of stationary sets for the class $\mathfrak{C}$. It is clear that if $\mathfrak{C}_2 \supset \mathfrak{C}_1$ then $S(\mathfrak{C}_1) \supset S(\mathfrak{C}_2)$. Consider now the chain

$$\mathfrak{D} \supset \mathfrak{D}\mathfrak{M} \supset \mathfrak{D}\mathfrak{B} \supset \mathfrak{D}\mathfrak{B}_1 \supset \mathfrak{D}\mathfrak{L} \supset \mathfrak{A}\mathfrak{L}$$

in which $\mathfrak{D}$ denotes the class of Darboux functions, $\mathfrak{D}\mathfrak{M}$ the class of Darboux measurable functions, $\mathfrak{D}\mathfrak{B}$ the class of Darboux Baire functions, $\mathfrak{D}\mathfrak{B}_1$ the class of Darboux Baire class 1 functions, $\mathfrak{D}\mathfrak{L}$ the class of Darboux lower semicontinuous functions, and $\mathfrak{A}\mathfrak{L}$ the class of approximately continuous lower semicontinuous functions. The collections $S(\mathfrak{D})$ and $S(\mathfrak{A}\mathfrak{L})$ have been characterized [1], [8]. In [5], Marcus found *sufficient* conditions for a set E to be in $S(\mathfrak{D}\mathfrak{B})$ and *necessary* conditions for a set to be in $S(\mathfrak{D}\mathfrak{M})$. He posed the problem of *characterizing* the stationary sets for the classes $\mathfrak{D}\mathfrak{B}$ and $\mathfrak{D}\mathfrak{M}$. In §3 we obtain these characterizations and in the process we characterize the stationary sets for the remaining classes in the above chain. Then in §4 we characterize the determining sets for these classes of functions.

2. **Preliminaries.** We shall be concerned with functions defined on an interval $[a, b]$. If f is such a function and A is a subset of $[a, b]$, then $f(A)$ denotes the set $\{y: f(x) = y \text{ for some } x \in A\}$. If g is defined on $[a, b]$ and f is defined on $g([a, b])$, $f \circ g$ denotes the function whose value at $x \in [a, b]$ is $f(g(x))$. By $\sim A$ we mean the complement of A in $[a, b]$, and $m_i(A)$ denotes the Lebesgue inner measure of A.

In the sequel we shall consider sets E which intersect every nonempty perfect subset of $[a, b]$. This is equivalent to $\sim E$ being totally

Received by the editors July 27, 1964.

[1] This author was supported by NSF Grant GP 1592.

[2] NSF Cooperative graduate fellow.

imperfect; that is, to $\sim E$ containing no nonempty perfect subsets. Every totally imperfect set has zero inner measure but there are totally imperfect sets with positive outer measure. In fact, the interval $[a, b]$ can be decomposed into two disjoint totally imperfect sets [**5**]. Each of these sets obviously has full outer measure.

A *Baire* function is a function f with the property that for every open set G, the set $\{x: f(x) \in G\}$ is a Borel set. A Baire class 1 function is a function which is the limit of a sequence of continuous functions. If f is a function with the property that $f(I)$ is connected for every interval $I \subset [a, b]$, then f is called a *Darboux* function. It is clear that the derivative of a differentiable function is a Darboux-Baire class 1 function.

The characterizations of $S(\mathfrak{D})$ and $S(\mathfrak{D}\mathfrak{M})$ assume the validity of the continuum hypothesis.

3. **Stationary sets.** We turn now to a discussion of the stationary sets for the various classes of functions under consideration. We begin with a general theorem.

THEOREM 1. *Let $\mathfrak{C}$ be a class of functions with the properties:*

(a) *if $f \in \mathfrak{C}$ and h is continuous from $[a, b]$ onto $[a, b]$, then $f \circ h \in \mathfrak{C}$;*

(b) *there exists a function $g \in \mathfrak{C}$ such that g equals a constant λ on a dense subset of $[a, b]$ but g is not identically equal to λ.*

Then a necessary condition that a set E be a stationary set for $\mathfrak{C}$ is that E intersect every nonempty perfect subset of $[a, b]$.

PROOF. Suppose E fails to intersect the nonempty perfect set $P' \subset [a, b]$. Let P be a nonempty nowhere dense subset of P'. Denote by N the dense set on which a function g of the hypothesis equals λ and let h be a continuous function from $[a, b]$ onto $[a, b]$, such that $h(\sim P) \subset N$. This is possible since N contains a denumerable dense subset which is of the same order type as the set of intervals contiguous to P. (See Goffman [**3**, p. 123].) (For example, h may be taken to be an appropriate Cantor-like function, continuous and nondecreasing, which is constant on each interval contiguous to P.) Consider now the function $f = g \circ h$. It follows from the hypothesis that $f \in \mathfrak{C}$. If $x \in \sim P$, then $h(x) \in N$, from which it follows that $f(x) = \lambda$. On the other hand, $\sim N$ is nonempty in $[a, b]$ and h is continuous; thus there is an $x \in P$ such that $h(x) \in \sim N$. For such an x, $f(x) \neq \lambda$. The function f establishes the necessity of the condition.

COROLLARY. *A necessary and sufficient condition that a set E be stationary for the class $\mathfrak{D}\mathfrak{L}$, $\mathfrak{D}\mathfrak{B}_1$ or $\mathfrak{D}\mathfrak{B}$ is that E intersect every nonempty perfect subset of $[a, b]$.*

PROOF. *Necessity.* The class $\mathfrak{DL}$ clearly satisfies condition (a) of Theorem 1. A construction[3] by Croft [3] of a function in $\mathfrak{DL}$, equal to zero almost everywhere but not identically zero, shows that (b) is satisfied. Hence the condition is necessary for $\mathfrak{DL}$, and thus for the larger classes $\mathfrak{DB}_1$ and $\mathfrak{DB}$.

Sufficiency. The sufficiency of the condition for the class $\mathfrak{DB}$ was shown in [5]. It therefore follows for the smaller classes $\mathfrak{DB}_1$ and $\mathfrak{DL}$.

It is possible for the class $\mathfrak{C}$ of Theorem 1 to be much smaller than $\mathfrak{DL}$. Indeed, given any function g satisfying (b), we may generate a smallest such class containing g by taking the composition of g with all continuous functions from $[a, b]$ onto $[a, b]$. Theorem 1 thus provides a necessary and sufficient condition for a set to be stationary for certain classes $\mathfrak{C}$ more restricted than $\mathfrak{DL}$ (the sufficiency being a consequence of the inclusion $\mathfrak{C}\subset\mathfrak{DL}$). Furthermore, if $\mathfrak{C}\subset\mathfrak{DL}$, then condition (b) *must* be satisfied in order to have $S(\mathfrak{C})=S(\mathfrak{DL})$. For if condition (b) is not satisfied then *any* dense subset of $[a, b]$ is stationary for $\mathfrak{C}$.

It is natural to ask whether there are any "important" classes $\mathfrak{C}$ properly contained in $\mathfrak{DL}$ such that $S(\mathfrak{C})=S(\mathfrak{DL})$. The answer depends, of course, on what classes one considers to be important. A natural candidate for such a class is the family $\mathfrak{AL}$ of approximately continuous lower semicontinuous functions. The collection $S(\mathfrak{AL})$ consists of all sets intersecting every set of positive inner measure. (This characterization follows readily from a similar characterization obtained by Marcus in [8]. See also Theorem 3, below.) Thus, under the assumption that the class $\mathfrak{AL}$ is the one which most naturally follows the class $\mathfrak{DL}$ in our chain, the above question is answered in the negative.

In the other direction, are there any classes $\mathfrak{C}$ properly containing $\mathfrak{DB}$ such that $S(\mathfrak{C})=S(\mathfrak{DB})$? There are classes $\mathfrak{C}$ which satisfy these requirements. For example, addition of a single suitable function to $\mathfrak{DB}$ gives a larger class $\mathfrak{C}$ such that $S(\mathfrak{C})=S(\mathfrak{DB})$. However, the *natural* class to consider here is the class $\mathfrak{DM}$. The following theorem characterizes the stationary sets for $\mathfrak{DM}$.

THEOREM 2. *A necessary and sufficient condition that the set E be stationary for the class $\mathfrak{DM}$ is that E intersect every nondenumerable measurable set.*

PROOF. *Necessity.* Assume that $\sim E$ contains a nondenumerable measurable subset M, and let M_0 be a nondenumerable zero measure subset of M. It follows from [1] that there exists a nonconstant Darboux function f, with f vanishing on $\sim M_0$, and hence on E. The func-

[3] The authors wish to thank the referee for calling their attention to this example.

tion f is thus equivalent to the zero function and therefore is measurable. This establishes the necessity of the condition.

Sufficiency. Let E be a set which intersects every nondenumerable measurable subset of $[a, b]$ and let f be a function in $\mathfrak{DM}$ which equals a constant λ on E. If for some $x_0 \in \sim E$, $f(x_0) \neq \lambda$, then the set $A = \{x : f(x) \neq \lambda\}$ is nondenumerable, since f is a Darboux function. Since f is measurable A is also measurable. But this implies that A intersects E, a contradiction.

The condition of measurability in the above theorem cannot be dropped, for there exist nondenumerable sets, all of whose measurable subsets are denumerable or finite [**4**, p. 168]. (This assumes the continuum hypothesis.)

4. **Determining sets.** A concept closely related to the concept of a stationary set is that of a determining set.

DEFINITION. Let $\mathfrak{C}$ be a class of functions defined on $[a, b]$. A set E is called a *determining set* for $\mathfrak{C}$ provided any two members of $\mathfrak{C}$ which agree on E must agree on all of $[a, b]$.

We denote by $D(\mathfrak{C})$ the determining sets for $\mathfrak{C}$. If $\mathfrak{C}_2 \supset \mathfrak{C}_1$ then $D(\mathfrak{C}_1) \supset D(\mathfrak{C}_2)$. It is clear that if $\mathfrak{C}$ contains the constant functions then $D(\mathfrak{C}) \subset S(\mathfrak{C})$, while if $\mathfrak{C}$ is closed under the operation of subtraction then $S(\mathfrak{C}) \subset D(\mathfrak{C})$. We observe that all of the classes considered in §3 contain the constants, but none of these classes is closed under subtraction. We proceed now to the characterization of the determining sets for the classes considered in §3.

THEOREM 3. (a) *The only determining set for the classes* $\mathfrak{D}$, $\mathfrak{DM}$, $\mathfrak{DB}$ *and* $\mathfrak{DB}_1$ *is the interval* $[a, b]$.

(b) *A necessary and sufficient condition that E be determining for* $\mathfrak{DL}$ *is that E intersect every nonempty perfect set.*

(c) *A necessary and sufficient condition that E be determining for* $\mathfrak{AL}$ *is that E intersect every set of positive inner measure.*

PROOF. (a) It suffices to observe that if $x_0 \in [a, b]$ then the functions

$$f(x) = \sin \frac{1}{x - x_0}, \quad f(x_0) = 1 \quad \text{and} \quad g(x) = \sin \frac{1}{x - x_0}, \quad g(x_0) = 0$$

are both in $\mathfrak{DB}_1$ and agree except at x_0.

(b) Necessity follows from the fact that the class $\mathfrak{DL}$ contains the constants, and that the condition is necessary for inclusion in the class $S(\mathfrak{DL})$.

Sufficiency. Suppose $f, g \in \mathfrak{DL}$ and for some $x_0 \in [a, b]$, $f(x_0) \neq g(x_0)$. Assume $f(x_0) > g(x_0)$ and let $3\epsilon = f(x_0) - g(x_0)$. Now g is lower semicontinuous and therefore in Baire class 1. Since g also satisfies the Darboux condition, there is a perfect nowhere dense set P containing x_0 such that the restriction of g to P is continuous at x_0 [**10**, p. 18]. Therefore there exists an interval (c, d) containing x_0, with $c, d \in \sim P$, such that if $x \in P' = P \cap (c, d)$, then $|g(x) - g(x_0)| < \epsilon$. On the other hand the semicontinuity of f allows us to determine an interval (c', d') containing x_0, with $c', d' \in \sim P'$, on which interval $f(x) > f(x_0) - \epsilon$. On the perfect set $P' \cap (c', d')$ we have $f(x) > f(x_0) - \epsilon > g(x_0) + \epsilon > g(x)$. We have shown that if two functions in $\mathfrak{D}\mathfrak{L}$ disagree at a point x_0, then they must disagree on a nonempty perfect set. It follows that any set which intersects every nonempty perfect subset of $[a, b]$ is determining for $\mathfrak{D}\mathfrak{L}$.

(c) It was shown in [6] that a set E is determining for the class of *bounded* functions in $\mathfrak{A}\mathfrak{L}$ if and only if E intersects every set of positive inner measure.

Since this class is contained in $\mathfrak{A}\mathfrak{L}$ the *necessity* of the condition for the class $\mathfrak{A}\mathfrak{L}$ is established. We turn now to a proof of the *sufficiency* of the condition. Let $f, g \in \mathfrak{A}\mathfrak{L}$. For each $N > 0$, define a function f_N by

$$f_N(x) = \begin{cases} N & \text{if } f(x) \geqq N, \\ f(x) & \text{if } -N < f(x) < N, \\ -N & \text{if } f(x) \leqq -N \end{cases}$$

and define the function g_N analogously. The functions f_N and g_N are bounded functions in $\mathfrak{A}\mathfrak{L}$. Now if $f(x_0) \neq g(x_0)$ and

$$N > \max[|f(x_0)|, |g(x_0)|],$$

then $f_N(x_0) \neq g_N(x_0)$. It follows from the above mentioned result of [6] that $\{x : f_N(x) \neq g_N(x)\}$ has positive inner measure. But this set is contained in the set $\{x : f(x) = g(x)\}$. The sufficiency is established.

The table below exhibits the characterizations of the stationary and determining sets for the function classes under consideration.

$\mathfrak{C}$	$S(\mathfrak{C})$	$D(\mathfrak{C})$
$\mathfrak{D}$	intersects every nondenumerable set	$[a, b]$
$\mathfrak{D}\mathfrak{M}$	intersects every nondenumerable measurable set	$[a, b]$
$\mathfrak{D}\mathfrak{B}$	intersects every nonempty perfect set	$[a, b]$
$\mathfrak{D}\mathfrak{B}_1$	intersects every nonempty perfect set	$[a, b]$
$\mathfrak{D}\mathfrak{L}$	intersects every nonempty perfect set	intersects every nonempty perfect set
$\mathfrak{A}\mathfrak{L}$	intersects every set of positive inner measure	intersects every set of positive inner measure

5. A comparison. We mentioned in §1 that every derivative is a Darboux function of Baire class 1. The converse is not valid. In order to give some sort of measure of how much more restrictive a condition it is for a function to be a derivative than it is for a function to be a Darboux function of Baire class 1, several interesting comparisons between these two classes of functions have been advanced [**9**], [**12**], [**13**]. S. Marcus proved [**6**] that a necessary and sufficient condition that E be stationary or determining for the class of (finite) derivatives is that E intersect every set of positive inner measure. This result, together with the results appearing in the row labeled by "$\mathfrak{D}\mathfrak{B}_1$" in the table above, provides still another comparison of the two classes.

References

1. N. Boboc and S. Marcus, *Sur la détermination d'une fonction par les valeurs prises sur un certain ensemble*, Ann. Sci. École Norm. Sup. (3) **76** (1959), 151–159.

2. A. Bruckner, *Stationary sets for certain classes of derivates of Darboux functions*, Michigan Math. J. **11** (1964), 305–309.

3. H. T. Croft, *A note on Darboux continuous functions*, J. London Math. Soc. **38** (1963), 9–10.

4. C. Goffman, *Real functions*, Rinehart, New York, 1960.

5. S. Marcus, *Les ensembles stationnaires de certaines classes de fonctions*, C. R. Acad. Sci. Paris **254** (1962), 1186–1188.

6. ———, *Sur une généralisation de la notion de quasi-analyticité*, C. R. Acad. Sci. Paris **254** (1962), 985–987.

7. ———, *Les ensembles stationnaires de certaines classes de fonctions dérivées*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. **32** (1962), 484–487.

8. ———, Private communication dated July 2, 1964.

9. I. Maximoff, *On continuous transformation of some functions into an ordinary derivative*, Ann. Scuola Norm. Sup. Pisa. **12** (1943), 147–160.

10. ———, *Sur les fonctions ayant la propriété de Darboux*, Prace Math.-Fiz. **43** (1936), 241–265.

11. I. Natanson, *Theory of functions of a real variable*, Vol. 2, Ungar, New York, 1960.

12. C. J. Neugebauer, *Darboux functions of Baire class one and derivatives*, Proc. Amer. Math. Soc. **13** (1962), 838–843.

13. Z. Zahorski, *Sur la première dérivée*, Trans. Amer. Math. Soc. **69** (1950), 1–54.

University of California, Santa Barbara