ON THE COMBINATORIAL SCHOENFLIES CONJECTURE

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Introduction. The combinatorial version of the Schoenflies conjecture in dimension n states: A combinatorial $(n-1)$-sphere on a combinatorial n-sphere decomposes the latter into two combinatorial n-cells. The cases $n=1$, 2 are obvious, the case $n=3$ was solved by J. W. Alexander [1], see also W. Graeub [5], and E. Moise [8]. Nothing is known for $n>3$ (compare J. F. Hudson and E. C. Zeeman in [6, p. 729]). On the other hand the following form of the Hauptvermutung for spheres is proved: If a combinatorial n-dimensional manifold is homeomorphic an n-dimensional sphere, then it is a combinatorial n-sphere if $n\neq 4$. This was proved by S. Smale for $n\neq 4$, 5, 7 in [9], and improved by E. C. Zeeman to $n\neq 4$ (unpublished). I would like to thank Professor Zeeman for pointing out this result to me. Further a generalized Schoenflies theorem was proved by M. Brown in [3]. If this Hauptvermutung were true for all n, a simple induction on the dimension n would prove the combinatorial Schoenflies conjecture for all n (compare Theorem 6). In the following we show that either the combinatorial Schoenflies conjecture is true for all dimensions n, or it is false for all dimensions $n>3$.

Notation. We are dealing with finite simplicial complexes, hereafter called complexes. An n-complex is a complex such that each simplex is a face of an n-dimensional simplex of the complex. If K, L are complexes then $K\cup L$, $K\cap L$, $K\cdot L$ denote the union, intersection, and the simplicial product of the complexes K and L. A combinatorial n-cell is a complex which has a subdivision isomorphic a subdivision of an n-simplex. And a combinatorial n-sphere is a complex which has a subdivision isomorphic to a subdivision of the boundary of an $(n+1)$-simplex. The link $L(s, K)$ of the simplex s of the complex K is the subcomplex consisting of those simplexes s' of K with s and s' are faces of a simplex in K and s' has no vertices in common with s. An n-dimensional combinatorial manifold is an n-complex such that the link of each vertex is a combinatorial $(n-1)$-sphere or a combinatorial $(n-1)$-cell. It is called closed, if its boundary is empty.

THEOREM 1 (J. W. ALEXANDER [2]). *Let S^n and S^k be combinatorial spheres of dimensions n and k, then the product $S^n\cdot S^k$ is a combinatorial $(n+k+1)$-sphere. Let S^n be a combinatorial n-sphere and E^k a com-*

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binatorial k-cell, then the product $S^n \cdot E^k$ is a combinatorial $(n+k+1)$-cell.

THEOREM 2 (J. W. ALEXANDER [2]). *Let S^n be a combinatorial n-sphere with the decomposition $S^n = E_1^n \cup E_2^n$, where E_1^n and E_2^n are n-complexes such that $E_1^n \cap E_2^n = S^{n-1}$ is a combinatorial $(n-1)$-sphere. If E_1^n is a combinatorial n-cell, then E_2^n is also a combinatorial n-cell.*

THEOREM 3 (H. KNESER [7]). *Let M^n, $n \geqq 1$, be a closed connected n-dimensional combinatorial manifold. Suppose $H_1(M^n; Z_2) = 0$ if $n \geqq 2$, where $H_1(M^n; Z_2)$ is the first homology group of M^n with coefficients in Z_2. Then any closed connected $(n-1)$-dimensional combinatorial manifold M^{n-1} ($M^0 = S^0$) on M^n determines a unique decomposition of M^n into two n-complexes K_1 and K_2 with $M^n = K_1 \cup K_2$ and $K_1 \cap K_2 = M^{n-1}$.*

THEOREM 4. *The combinatorial Schoenflies conjecture is true in dimension n if and only if a combinatorial n-sphere S^n on a combinatorial $(n+1)$-sphere S^{n+1} decomposes the latter into two combinatorial manifolds E_1^{n+1} and E_2^{n+1} with $E_1^{n+1} \cup E_2^{n+1} = S^{n+1}$ and $E_1^{n+1} \cap E_2^{n+1} = S^n$.*

PROOF. The "if" part: Let S^{n-1} be a combinatorial $(n-1)$-sphere on the combinatorial n-sphere S^n. From Theorem 3 we have the decomposition of S^n into two n-complexes E_1^n and E_2^n with $E_1^n \cup E_2^n = S^n$ and $E_1^n \cap E_2^n = S^{n-1}$. Let $S^0 = \{x, y\}$ be a 0-sphere. We form the simplicial product $S^n \cdot S^0 = E_1^n \cdot S^0 \cup E_2^n \cdot S^0$. Of course $E_1^n \cdot S^0 \cap E_2^n \cdot S^0 = S^{n-1} \cdot S^0$. But $S^{n-1} \cdot S^0$ is now a combinatorial n-sphere on the combinatorial $(n+1)$-sphere $S^n \cdot S^0$. And $E_1^n \cdot S^0$ and $E_2^n \cdot S^0$ are then by hypothesis combinatorial manifolds. We compute the following links: $L(x, E_1^n \cdot S^0) = E_1^n$ and $L(x, E_2^n \cdot S^0) = E_2^n$. E_1^n and E_2^n must be n-cells.

The "only if" part: From Theorem 3 we have that S^{n+1} is decomposed into two $(n+1)$-complexes E_1^{n+1} and E_2^{n+1} with $S^{n+1} = E_1^{n+1} \cup E_2^{n+1}$ and $E_1^{n+1} \cap E_2^{n+1} = S^n$. To show that these $(n+1)$-complexes are combinatorial manifolds, we have only to check that the links of their vertices are either combinatorial n-spheres or combinatorial n-cells. For a vertex e not in S^n we have $L(e, E_i^{n+1}) = L(e, S^{n+1})$, $i = 1, 2$, and this link is therefore a combinatorial n-sphere. If the vertex e lies on S^n then the combinatorial $(n-1)$-sphere $L(e, S^n)$ decomposes the combinatorial n-sphere $L(e, S^{n+1})$ by hypothesis into two combinatorial n-cells, which lie in E_1^{n+1} and E_2^{n+1}. These n-cells are the links of e in E_1^{n+1} and E_2^{n+1}.

COROLLARY 1. *If the combinatorial Schoenflies conjecture is true in dimension n, then it is also true in dimension $n-1$, and therefore in all dimensions up to n.*

THEOREM 5 (M. BROWN [3] AND [4]). *Let S^n be a combinatorial n-sphere and S^{n-1} a combinatorial $(n-1)$-sphere on S^n. Then S^{n-1} decomposes S^n into two complexes E_1^n and E_2^n with $S^n = E_1^n \cup E_2^n$, $E_1^n \cap E_2^n = S^{n-1}$, and E_1^n, E_2^n are topological n-cells, i.e. E_1^n and E_2^n are homeomorphic to an n-simplex.*

PROOF. S^{n-1} is "bi-collared" in S^n, and the generalized Schoenflies theorem of [3] can be applied (see [4]). By the *combinatorial Hauptvermutung* for n-spheres we mean the statement, that if an n-dimensional combinatorial manifold is homeomorphic to an n-sphere then it is a combinatorial n-sphere.

THEOREM 6. *If the combinatorial Schoenflies conjecture is true in dimension $n-1$, and if the combinatorial Hauptvermutung for n-spheres is true, then the combinatorial Schoenflies conjecture is true in dimension n.*

PROOF. Suppose S^{n-1} is a combinatorial $(n-1)$-sphere on the combinatorial n-sphere S^n. From Theorem 4 we have the decomposition $S^n = E_1^n \cup E_2^n$, where E_1^n and E_2^n are n-dimensional combinatorial manifolds, and $E_1^n \cap E_2^n = S^{n-1}$. By Theorem 5 we know that E_1^n and E_2^n are topological n-cells. We consider the n-dimensional combinatorial manifold $\tilde{S}^n = E_1^n \cup S^{n-1} \cdot E^0$, where $E^0 = \{x\}$ is a 0-cell. Since E_1^n and $S^{n-1} \cdot E^0$ are topological n-cells, $\tilde{S}^n$ is homeomorphic to the n-sphere, and by hypothesis $\tilde{S}^n$ is a combinatorial n-sphere. Since $S_1^{n-1} \cdot E^0$ is a combinatorial n-cell, we can apply Theorem 2 and conclude that E_1^n is also a combinatorial n-cell. And by the same argument E_2^n is a combinatorial n-cell too.

THEOREM 7 (S. SMALE [9] AND E. C. ZEEMAN). *The combinatorial Hauptvermutung for n-spheres is true for $n \neq 4$.*

COROLLARY 2. *Either the combinatorial Schoenflies conjecture is true for all dimensions n, or it is false for all dimensions $n > 3$.*

COROLLARY 3. *If the combinatorial Schoenflies conjecture can be proved for one dimension $n_0 > 3$, then it is true for all dimensions n. If a counterexample to the combinatorial Schoenflies conjecture can be found in one dimension $n_0 > 3$, then the conjecture is false for all dimensions $n > 3$, and subsequently the combinatorial Hauptvermutung for 4-spheres would be false.*

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