SYMMETRY FOR THE ENVELOPING ALGEBRA OF
A RESTRICTED LIE ALGEBRA

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In a recent paper [1], Berkson has shown that the restricted enveloping algebra \( U \) of a restricted finite-dimensional Lie algebra \( L \) is a Frobenius algebra. By requiring that each transformation in the adjoint representation of \( L \) have zero trace (a condition satisfied by any nilpotent \( L \) or any \( L \) for which \( [L, L] = L \)) it turns out that \( U \) is actually symmetric. A proof of this is given below.

We let \( L \) be a restricted Lie algebra which is finite-dimensional over a field \( K \) of characteristic \( p > 0 \). For \( x \in L \) let \( D_x \) be defined on \( L \) by \( D_x y = [x, y] \), and let \( \text{Tr}(D_x) \) denote the trace of \( D_x \). \( U \) will denote the restricted enveloping algebra of \( L \) as defined and discussed in [2, pp. 185–192], and \( U^* \) denotes the dual space of \( U \) over \( K \). For \( u \in U \) and \( \phi \in U^* \) define \( u\phi \) and \( \phi u \) by \( (u\phi)(v) = \phi(vu) \), \( (\phi u)(v) = \phi(uv) \) for all \( v \in U \). We choose a fixed ordered basis \( x_1, \ldots, x_n \) of \( L \) and thus \( \{x_1^i \cdot \cdots \cdot x_n^j : 0 \leq i_j \leq p-1 \} \) is a basis of \( U \). For each such basis element of \( U \) we define the degree as \( \sum i_j \) and for a linear combination of basis elements define the degree as the maximum of the degrees of basis elements which appear with nonzero coefficients. Let \( \phi_0 \) be defined as the linear functional on \( U \) which vanishes at each basis element except that \( \phi_0(x_1^{p-1} \cdot \cdots \cdot x_n^{p-1}) = 1 \). The main result of [1] is that the linear mapping \( u \to \phi_0 u \) from \( U \) to \( U^* \) is one-one and onto. The result to be proved here is the following:

**Theorem.** \( u\phi_0 = \phi_0 u \) for all \( u \in U \) iff \( \text{Tr}(D_x) = 0 \) for all \( x \in L \). Thus, if the latter condition is satisfied, \( U \) is symmetric, i.e., the bilinear form \( (u, v) = \phi_0(uv) \) is symmetric, nondegenerate, and \( (uv, w) = (u, vw) \) for all \( u, v, w \) in \( U \).

The proof of the theorem will follow from several lemmas.

**Lemma 1.** Suppose \( m \leq n(p-1) \) and \( y_1, \ldots, y_m \in L \). Then \( \phi_0(y_1, \ldots, y_m) = \phi_0(y_{i_1}, \ldots, y_{i_m}) \) for any permutation \( i_1, \ldots, i_m \) of 1, \( \ldots, m \). If \( m < n(p-1) \) then \( \phi_0(y_1, \ldots, y_m) = 0 \).

**Proof.** By using techniques like those used in [2] it follows that the degree of \( y_1, \ldots, y_m \) is no greater than \( m \) and that \( y_1, \ldots, y_m = y_{i_1} \cdot \cdots \cdot y_{i_m} + v \) where \( v \) has degree less than \( m \).

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Lemma 2. For \( u, v \in U \) let \([u, v] = uv - vu\). Then for \( 0 \leq m < p \) and \( x, y \in L \),
\[
[x, y^m] = \sum_{i=0}^{m} C_i (-1)^{i} y^{m-i} D^i y(x).
\]

Proof. The proof is by induction on \( m \). The case \( m = 1 \) is immediate; we assume the result as stated to prove it for \( m + 1 \). Now \([x, y^{m+1}] = [x, y^m y] = y^m [x, y] + [x, y^m] y = - y^m D_x x + y [x, y^m] - [y, [x, y^m]]\). If the induction hypothesis is used on each of the last two terms, together with \([y, y^{m-k} D^k x] = y^{m-k} D^{k+1} x\), a straightforward computation will give the desired conclusion.

Lemma 3. Let \( u_0 = x_1^{p-1} \cdots x_n^{p-1} \). For \( x \in L \) we have \( \phi_0(u_0 x) = \phi_0(x u_0) + \text{Tr}(D_x) \).

Proof. Let \( D_x x_i = \sum \lambda_{ij} x_j \). By virtue of Lemma 2, \( x_i^{p-1} x = x x_i^{p-1} + x_i^{p-2} [x, x_i] + u_i \) where \( u_i \) has degree less than \( p - 1 \). From Lemma 1 we obtain \( \phi_0(x_1^{p-1} \cdots x_i^{p-1} x \cdots x_n^{p-1}) = \phi_0(x_1^{p-1} \cdots x_i^{p-1} \cdots x_n^{p-1}) + \sum \lambda_{ij} \phi_0(x_1^{p-1} \cdots x_i^{p-2} x_{ij} \cdots x_n^{p-1}) \). However, since \( x_i \in L \), each of the terms in the last summation is zero for \( j \neq i \). Thus the sum reduces to \( \lambda_{ii} \phi_0(u_0) = \lambda_{ii} \). An induction argument can then be used to conclude that \( \phi_0(u_0 x) = \phi_0(x u_0) + \sum \lambda_{ii} = \phi_0(x u_0) + \text{Tr}(D_x) \).

Proof of the theorem. For each \( u \in U \) there is a unique \( u^* \in U \) such that \( u^* \phi_0 = \phi_0 u \). The mapping \( u \mapsto u^* \) is clearly linear and is one-one for if \( u^* \) is zero then \( (\phi_0)(u) = 0 \) for all \( v \in U \) and this implies \( u = 0 \). Moreover, it is an automorphism of the associative algebra \( U \) since \( (uv)^* \phi_0(w) = \phi_0(uvw) = \phi_0(vuw^*) = \phi_0(wu^{*}v^*) = (u^*v^*) \phi_0(w) \) for all \( w \). Suppose \( \text{Tr}(D_x) = 0 \) for all \( x \in L \). Then \( \phi_0(x u_0) = \phi_0(u_0 x) \). From Lemma 1 we have \( \phi_0(x u) = \phi_0(u x) \) for any basis element \( u \) of smaller degree than \( n(p - 1) \). Hence the same equation holds for all \( u \) and this implies \( x^* = x \) for all \( x \in L \). Since \( U \) is generated by \( 1 \) and \( L \) we have \( u^* = u \) for all \( u \in U \).

Conversely, if \( u_0 = \phi_0 u \) for all \( u \) then \( x = x^* \) for all \( x \in L \) and Lemma 3 shows that \( \text{Tr}(D_x) = 0 \).

References


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