SYMMETRY FOR THE ENVELOPING ALGEBRA OF A RESTRICTED LIE ALGEBRA

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In a recent paper [1], Berkson has shown that the restricted enveloping algebra U of a restricted finite-dimensional Lie algebra L is a Frobenius algebra. By requiring that each transformation in the adjoint representation of L have zero trace (a condition satisfied by any nilpotent L or any L for which $[L, L]=L$) it turns out that U is actually symmetric. A proof of this is given below.

We let L be a restricted Lie algebra which is finite-dimensional over a field K of characteristic $p>0$. For $x\in L$ let D_x be defined on L by $D_xy=[x, y]$, and let $\mathrm{Tr}(D_x)$ denote the trace of D_x. U will denote the restricted enveloping algebra of L as defined and discussed in [2, pp. 185–192], and U^* denotes the dual space of U over K. For $u\in U$ and $\phi\in U^*$ define $u\phi$ and ϕu by $(u\phi)(v)=\phi(vu)$, $(\phi u)(v)=\phi(uv)$ for all $v\in U$. We choose a fixed ordered basis $x_1, \cdots, x_n$ of L and thus $\{x_1^{i_1}\cdots x_n^{i_n}: 0\leqq i_j\leqq p-1\}$ is a basis of U. For each such basis element of U we define the degree as $\sum i_j$ and for a linear combination of basis elements define the degree as the maximum of the degrees of basis elements which appear with nonzero coefficients. Let ϕ_0 be defined as the linear functional on U which vanishes at each basis element except that $\phi_0(x_1^{p-1}\cdots x_n^{p-1})=1$. The main result of [1] is that the linear mapping $u\to u\phi_0$ from U to U^* is one-one and onto. The result to be proved here is the following:

THEOREM. *$u\phi_0=\phi_0u$ for all $u\in U$ iff $\mathrm{Tr}(D_x)=0$ for all $x\in L$. Thus, if the latter condition is satisfied, U is symmetric, i.e., the bilinear form $(u, v)=\phi_0(uv)$ is symmetric, nondegenerate, and $(uv, w)=(u, vw)$ for all u, v, w in U.*

The proof of the theorem will follow from several lemmas.

LEMMA 1. *Suppose $m\leqq n(p-1)$ and $y_1, \cdots, y_m\in L$. Then $\phi_0(y_1, \cdots, y_m)=\phi_0(y_{i_1}, \cdots, y_{i_m})$ for any permutation $i_1, \cdots, i_m$ of $1, \cdots, m$. If $m<n(p-1)$ then $\phi_0(y_1, \cdots, y_m)=0$.*

PROOF. By using techniques like those used in [2] it follows that the degree of $y_1, \cdots, y_m$ is no greater than m and that $y_1, \cdots, y_m=y_{i_1}\cdots y_{i_m}+v$ where v has degree less than m.

Received by the editors September 8, 1964.

LEMMA 2. *For u, v in U let* $[u, v]=uv-vu$. *Then for* $0\leqq m<p$ *and* x, y *in* L, $[x, y^m]=\sum_1^m {}_mC_k(-1)^k y^{m-k}D_y^k(x)$.

PROOF. The proof is by induction on m. The case $m=1$ is immediate; we assume the result as stated to prove it for $m+1$. Now $[x, y^{m+1}]=[x, y^m y] = y^m[x, y] + [x, y^m]y = -y^m D_y x + y[x, y^m] - [y, [x, y^m]]$. If the induction hypothesis is used on each of the last two terms, together with $[y, y^{m-k}D^k x]=y^{m-k}D_y^{k+1}x$, a straightforward computation will give the desired conclusion.

LEMMA 3. *Let* $u_0=x_1^{p-1}\cdots x_n^{p-1}$. *For* $x\in L$ *we have* $\phi_0(u_0x)=\phi_0(xu_0)+\mathrm{Tr}(D_x)$.

PROOF. Let $D_x x_i=\sum\lambda_{ji}x_j$. By virtue of Lemma 2, $x_i^{p-1}x=xx_i^{p-1}+x_i^{p-2}[x, x_i]+u_i$ where u_i has degree less than $p-1$. From Lemma 1 we obtain $\phi_0(x_1^{p-1}\cdots x_i^{p-1}x\cdots x_n^{p-1})=\phi_0(x_1^{p-1}\cdots xx_i^{p-1}\cdots x_n^{p-1})+\sum\lambda_{ji}\phi_0(x_1^{p-1}\cdots x_i^{p-2}x_j\cdots x_n^{p-1}$. However, since $x_j^p\in L$, each of the terms in the last summation is zero for $j\neq i$. Thus the sum reduces to $\lambda_{ii}\phi_0(u_0)=\lambda_{ii}$. An induction argument can then be used to conclude that $\phi_0(u_0x)=\phi_0(xu_0)+\sum\lambda_{ii}=\phi_0(xu_0)+\mathrm{Tr}(D_x)$.

PROOF OF THE THEOREM. For each $u\in U$ there is a unique $u^*\in U$ such that $u^*\phi_0=\phi_0u$. The mapping $u\to u^*$ is clearly linear and is one-one for if u^* is zero then $(v\phi_0)(u)=0$ for all $v\in U$ and this implies $u=0$. Moreover, it is an automorphism of the associative algebra U since $(uv)^*\phi_0(w)=\phi_0(uvw)=\phi_0(vwu^*)=\phi_0(wu^*v^*)=(u^*v^*)\phi_0(w)$ for all w implies $(uv)^*=u^*v^*$.

Suppose $\mathrm{Tr}(D_x)=0$ for all $x\in L$. Then $\phi_0(xu_0)=\phi_0(u_0x)$. From Lemma 1 we have $\phi_0(xu)=\phi_0(ux)$ for any basis element u of smaller degree than $n(p-1)$. Hence the same equation holds for all u and this implies $x^*=x$ for all $x\in L$. Since U is generated by 1 and L we have $u^*=u$ for all $u\in U$.

Conversely, if $u\phi_0=\phi_0u$ for all u then $x=x^*$ for all $x\in L$ and Lemma 3 shows that $\mathrm{Tr}(D_x)=0$.

REFERENCES

1. Astrid J. Berkson, *The u-algebra of a restricted Lie algebra is Frobenius*, Proc. Amer. Math. Soc. **15** (1964), 14–15.

2. Nathan Jacobson, *Lie algebras*, Interscience, New York, 1962.

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