

ON A CLASS OF HOLOMORPHIC FUNCTIONS¹

NICOLAS ARTÉMIADIS

I. Introduction. In Part I of this paper, some inequalities, concerning functions considered in Theorem 2, are obtained. In Part II we introduce the class A_p . A function $F(z) = z + \sum a_n z^n$, ($z = re^{it}$), belongs to A_p if it is holomorphic in $|z| < 1$, if the $\{a_n\}$ is real and if there exist a non-negative integer p and real numbers r_p, B_p such that:

$$\inf_{t,r} \{ \sin t \cdot I[F(z)/(1-z)^p] \} = B_p \quad (t = \text{real}, 0 \leq r_p \leq r < 1).$$

For $p = r_p = B_p = 0$ we get the class \mathcal{C} of typically real functions [3].

Theorems concerning the coefficients $\{a_n\}$ of $F(z)$, tauberian theorems and summability methods for $\sum a_n$ are obtained.

We denote by $T(f) = \phi(t) = \int_{-\infty}^{\infty} f(x)e^{itz} dx$, the Fourier transform of $f \in L_1$.

THEOREM a [1, p. 20]. *If $f \in L_1$, $|f(x)| \leq M$ in $-h \leq x \leq h$, $h > 0$, and $\phi(t) \geq 0$, then $\phi \in L_1$.*

The above theorem can be easily generalized as follows:

THEOREM 1. *If $f \in L_1$, $|f(x)| \leq M$ in $-h \leq x \leq h$, $h > 0$ and if $\alpha \leq \arg \phi(t) \leq \alpha + (\pi/2)$, then $\phi \in L_1$.*

PROOF. We may assume $\alpha = 0$. If $\alpha \neq 0$ we consider the function $f_\alpha(x) = f(x)e^{-i\alpha x}$ for which $0 \leq \arg T(f_\alpha) \leq \pi/2$. Next put

$$F(x) = [f(x) + \overline{f(-x)}]/2, \quad G(x) = [f(x) - \overline{f(-x)}]/2i.$$

We have

$$T(F) = R_e T(f) \geq 0, \quad T(G) = IT(f) \geq 0.$$

It follows from Theorem a that $R_e T(f) \in L_1$, $IT(f) \in L_1$. Therefore $\phi \in L_1$.

Notice that since $R_e T(f)$, $IT(f)$ both belong to L_1 , the inversion holds, so that

$$f(x) + \overline{f(-x)} = (1/\pi) \int_{-\infty}^{\infty} R_e T(f) e^{-itz} dt \quad \text{a.e.,}$$

Presented to the Society, August 27, 1964; received by the editors April 8, 1964 and, in revised form, July 1, 1964.

¹ Part of this research has been done under a grant of the University of Wisconsin Alumni Research Foundation during the summer of 1963.

$$f(x) - \overline{f(-x)} = (i/\pi) \int_{-\infty}^{\infty} IT(f)e^{-itx} dt \quad \text{a.e.}$$

THEOREM 2. *Hypothesis.* $f \in L_1$; $f(x) = 0$ for $x < 0$; put $\psi(t) = \int_0^\infty f(x) \sin tx \, dx$, $\sup_t R_e[t\psi(t)] = A$, $\inf_t R_e[t\psi(t)] = B$ and suppose A, B finite.

Conclusion. For $x \geq 0$

- (a) If $A = 0$ then $2R_e \int_0^\infty f(y) dy \leq R_e \int_0^x f(y) dy \leq 0$.
- (b) If $B = 0$ then $0 \leq R_e \int_0^x f(y) dy \leq 2R_e \int_0^\infty f(y) dy$.
- (c) If $A \neq 0$ then $R_e \int_0^x f(y) dy \leq Ax$.
- (d) If $B \neq 0$ then $R_e \int_0^x f(y) dy \geq Bx$.

PROOF. Consider the functions

$$\begin{aligned} f_1(x) &= (1/2i)[f(x) - f(-x)], \\ g(x) &= \begin{cases} e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases} \quad \lambda > 0, \\ g_1(x) &= (1/2i)[g(x) - g(-x)]. \end{aligned}$$

We find:

$$T(g_1) = t/(t^2 + \lambda^2), \quad T(f_1) = \psi(t), \quad T[Ae^{-\lambda|x|}/2\lambda] = A/(\lambda^2 + t^2).$$

Also

$$(2.1) \quad \begin{aligned} R_e T[(Ae^{-\lambda|x|}/2\lambda) - f_1 * g_1] &= R_e [(A - t\psi(t))/(t^2 + \lambda^2)] \\ &= [A - R_e t\psi(t)]/(t^2 + \lambda^2) > 0, \end{aligned}$$

$$(2.2) \quad \begin{aligned} R_e T[f_1 * g_1 - (Be^{-\lambda|x|}/2\lambda)] &= R_e [(t\psi(t) - B)/(t^2 + \lambda^2)] \\ &= [R_e t\psi(t) - B]/(t^2 + \lambda^2) \geq 0. \end{aligned}$$

We have:

$$(2.3) \quad f_1 * g_1 = (1/4) \int_0^\infty [f(x+y) - f(-y-x) - f(x-y) + f(y-x)] e^{-\lambda y} dy.$$

It follows from (2.1), (2.2) that the functions:

$$(Ae^{-\lambda|x|}/2\lambda) - R_e[f_1 * g_1], \quad R_e[f_1 * g_1] - (Be^{-\lambda|x|}/2\lambda)$$

are bounded, continuous and they have non-negative Fourier transforms, which, by Theorem a, belong to L_1 . Therefore the inversion formula holds everywhere:

$$\frac{Ae^{-\lambda|x|}}{2\lambda} - R_e[f_1 * g_1] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A - R_e t\psi(t)}{t^2 + \lambda^2} e^{-izt} dt \quad \text{everywhere}$$

$$R_e[f_1 * g_1] - \frac{Be^{-\lambda|x|}}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{R_e t \psi(t) - B}{t^2 + \lambda^2} e^{-ixt} dt \quad \text{everywhere.}$$

By taking the absolute values of both members we get:

$$(2.4) \quad \left| (Ae^{-\lambda|x|}/2\lambda) - R_e[f_1 * g_1] \right| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A - R_e t \psi(t)}{t^2 + \lambda^2} dt$$

$$= (A/2\lambda) - R_e[f_1 * g_1]_{x=0},$$

$$(2.5) \quad \left| R_e[f_1 * g_1] - (Be^{-\lambda|x|}/2\lambda) \right| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{R_e t \psi(t) - B}{t^2 + \lambda^2} dt$$

$$= R_e[f_1 * g_1]_{x=0} - (B/2\lambda).$$

For $x=0$, we get from (2.3):

$$R_e[f_1 * g_1]_{x=0} = (1/2)R_e \int_0^{\infty} f(y)e^{-\lambda y} dy.$$

Suppose $A=0$. Then (2.4) becomes:

$$\left| R_e[f_1 * g_1] \right| \leq -R_e[f_1 * g_1]_{x=0}$$

or

$$\left| R_e(1/4) \int_0^{\infty} [f(x+y) - f(-y-x) - f(x-y) + f(y-x)]e^{-\lambda y} dy \right|$$

$$\leq - (1/2)R_e \int_0^{\infty} f(y)e^{-\lambda y} dy.$$

The conclusion (a) follows if in the last inequality we let $\lambda \rightarrow 0+$. If $B=0$, then conclusion (b) follows from (2.5) in the same way. Suppose now $A \neq 0$. We get from (2.4):

$$\frac{Ae^{-\lambda|x|} - A}{2\lambda} - R_e \frac{1}{4} \int_0^{\infty} [f(x+y) - f(-y-x) - f(x-y) + f(y-x)]e^{-\lambda y} dy$$

$$\leq -R_e \frac{1}{2} \int_0^{\infty} f(y)e^{-\lambda y} dy$$

and the conclusion (c) follows if $\lambda \rightarrow 0+$. If $B \neq 0$, conclusion (d) follows from (2.5) in a similar way.

COROLLARY. *Hypothesis.* $f \in L_1$; $f(x) = 0$ for $x < 0$; put $\psi(t) = \int_0^{\infty} f(x) \sin tx dx$, $\sup_t I[t\psi(t)] = A^*$, $\inf_t I[t\psi(t)] = B^*$, and suppose A^* , B^* finite.

Conclusion. The conclusion of Theorem 2 holds if we replace A , B , R_e by A^* , B^* , I respectively.

PROOF. Put $\psi^*(t) = \int_0^\infty (-i)f(x) \sin tx \, dx$. We have $I[t\psi(t)] = R_c[t\psi^*(t)]$ and the corollary follows if in the conclusion of Theorem 2 we replace f by $-if$.

II. **Definition of the class \mathcal{C} of typically real functions [3].** A function $F(z) = z + \sum_{n=2}^\infty a_n z^n$ ($z = re^{it}$) is said to be typically real in the circle $|z| < 1$, if it is holomorphic in this circle and if $F(z)$ is real for real values of z , but for no other values in $|z| < 1$.

It follows from the above definition that the coefficients $\{a_n\}$ are real. Also, one can prove, that $F \in \mathcal{C}$ if and only if, $\text{sign } IF(z) = \text{sign } Iz$. This last relation is equivalent to: $\sin t \cdot IF(re^{it}) \geq 0$.

As we mentioned in the introduction of this paper, the class A_p is a generalization of the class \mathcal{C} . More precisely, we notice that \mathcal{C} is a proper subclass of A_0 . In fact, $F(z) = [z/(1-z)] + 2z^2$ belongs to A_0 but not to \mathcal{C} .

Also, if $F \in A_p$, we have for $r < 1$ and $t = 0$, $\sin t \cdot I[F(z)/(1-z)^p] = 0$; therefore $B_p \leq 0$.

THEOREM 3. *If $F \in A_0$, then:*

- (a) $|a_{n+1} - a_{n-1}| \leq 2 - 4B_0$,
- (b) $|a_n| \leq n(1 - 2B_0)$,
- (c) $1 + a_2 + \dots + a_{n-1} + (a_n/2) \geq B_0 n$

where $n = 1, 2, 3, \dots, a_0 = 0$.

PROOF. Put:

$$f(x) = \begin{cases} a_n r^n + a_{n+1} r^{n+1} & \text{for } n \leq x < n + 1, \\ 0 & \text{for } x < 0, \end{cases}$$

$$n = 0, 1, 2, \dots, 0 \leq r_0 \leq r < 1.$$

$$\psi(t) = \int_0^\infty f(x) \sin tx \, dx.$$

We find:

$$\begin{aligned} t\psi(t) &= 2 \sin t \cdot IF(z) \\ &= [r + a_2 r^2 \cos t + (a_3 r^3 - r) \cos 2t + (a_4 r^4 - a_2 r^2) \cos 3t + \dots] \\ &\geq 2B_0. \end{aligned}$$

Multiplying both sides by $1 \pm \cos nt$ and integrating from 0 to 2π we get the inequality:

$$\pi r [2 \pm (a_{n+1} - a_{n-1} r^{-2}) r^{n-1}] \geq 2B_0 \cdot 2\pi$$

or $|a_{n+1} - a_{n-1}| \leq 2 - 4B_0$, ($n \geq 1$). Next put $B_r = \inf_t [t\psi(t)]$. We have

$B_r \geq 2B_0$. The function f satisfies the assumptions of Theorem 2, therefore $r + a_2 r^2 + \cdots + a_{n-1} r^{n-1} + (a_n r^n / 2) \geq B_r n / 2 \geq B_0 n$. The conclusion (c) follows if we let $r \rightarrow 1 -$.

Note. For $r_0 = B_0 = 0$ we get the well-known inequalities: $|a_n| \leq n$, $1 + a_2 + \cdots + a_{n-1} + (a_n / 2) \geq 0$ for the functions of the class \mathcal{C} ([2], [3]).

An analogous theorem can be given for functions of the class A_p ($p \geq 1$).

Put $s_n^{(0)} = a_n$, $s_n^{(p)} = \sum_{i=1}^n s_i^{(p-1)}$ ($p = 1, 2, \cdots$). It is easy to see that if $F \in A_p$ then $F_p(z) = \sum_{n=1}^{\infty} s_n^{(p)} z^n$ belongs to A_0 . Applying Theorem 3 to F_p we get:

THEOREM 3*. If $F \in A_p$ then

- (a) $|s_{n+1}^{(p)} - s_{n-1}^{(p)}| \leq 2 - 4B_p$,
 (b) $|s_n^{(p)}| \leq n(1 - 2B_p)$ ($n \geq 1$, $s_0^{(p)} = 0$),
 (c) $1 + s_2^{(p)} + s_3^{(p)} + \cdots + s_{n-1}^{(p)} + (s_n^{(p)} / 2) \geq B_p n$.

THEOREM 4. If $F \in A_0$ and $F(r) \sim (1-r)^{-1}$ then

$$\sum_{k=1}^n (s_k^{(1)} / k) \sim n \quad (n \rightarrow \infty).$$

PROOF. By (c) of Theorem 3 we have: $(s_n^{(1)} / n) - (a_n / 2n) - B_0 \geq 0$. Also

$$\sum_{n=1}^{\infty} [(s_n^{(1)} / n) - (a_n / 2n) - B_0] r^n \sim [(1 - B_0) / (1 - r)].$$

It follows from Hardy-Littlewood's theorem [4, p. 226]:

$$\sum_{k=1}^n [(s_k^{(1)} / k) - (a_k / 2k) - B_0] \sim n(1 - B_0) \quad (n \rightarrow \infty)$$

or

$$(*) \quad \sum_{k=1}^n [(s_k^{(1)} / k) - (a_k / 2k)] \sim n \quad (n \rightarrow \infty).$$

By (a) of Theorem 3 we have:

$$2 - 4B_0 + a_{n+1} - a_{n-1} \geq 0,$$

$$\sum_{n=1}^{\infty} (2 - 4B_0 + a_{n+1} - a_{n-1}) r^n \sim [(2 - 4B_0) / (1 - r)].$$

Applying again Hardy-Littlewood's theorem we get:

$$\lim_{n \rightarrow +\infty} [(a_n + a_{n+1})/n] = 0.$$

We write:

$$\frac{a_{n+1} + a_{n+2}}{n} = \frac{n + 1}{n} \left(\frac{a_{n+1}}{n + 1} + \frac{a_{n+2}}{n + 2} \right) + \frac{a_{n+2}}{n(n + 2)}.$$

Since $|a_n| \leq n(1 - 2B_0)$ we have $\lim_{n \rightarrow \infty} [a_{n+2}/n(n+2)] = 0$. Therefore $\lim_{n \rightarrow \infty} [(a_{n+1} + a_{n+2})/n] = 0 = \lim_{n \rightarrow \infty} [(a_{n+1}/n+1) + (a_{n+2}/n+2)]$,

$$1 + (a_2/2) + (a_3/3) + \dots + (a_n/n) = 0(n) \quad (n \rightarrow +\infty),$$

and the conclusion of the theorem follows from (*).

To generalize Theorem 4 we state the following:

THEOREM b [4, Ex. 8, p. 242]. *If $a_n \geq 0$ and $(\sum a_n x^n) \sim (1-x)^{-\alpha}$, $(\alpha > 1)$ then $\sum_{k=1}^n a_k \sim (n^\alpha/\Gamma(\alpha+1))$, $(n \rightarrow \infty)$.*

THEOREM 4*. *If $F \in A_p$ and $F(r) \sim (1-r)^{-\alpha}$, $(\alpha > 1)$ then $\sum_{k=1}^n (s_k^{p+1}/k) \sim (n^{\alpha+p}/\Gamma(\alpha+p+1))$, $(n \rightarrow \infty)$.*

PROOF. Put $F_p(x) = \sum_{n=1}^\infty s_n^{(p)} z^n$. Notice that $F_p \in A_0$, $F_p(r) \sim (1-r)^{-(\alpha+p)}$ and use Theorems 3 and b. The proof is very similar to the proof of Theorem 4.

The following theorem provides a summability method for $\sum a_n$, where $F(z) = \sum a_n z^n$ belongs to A_p . A similar theorem is given in [2] for the class \mathcal{C} .

THEOREM 5. *If $F \in A_p$, $(p \geq 1)$ and $\lim_{r \rightarrow 1^-} F(r) = F(1)$ exists and is positive then:*

$$\lim_{n \rightarrow \infty} \frac{\Gamma(p + 1)}{n^{p+1}} [1 + s_2^{(p+1)} + \dots + s_{n-1}^{(p+1)} + (s_n^{(p+1)}/2)] = F(1).$$

PROOF. From (c) of Theorem 3* it follows that:

$$k_n = s_n^{(p+1)} - (s_n^{(p)}/2) - B_p n \geq 0.$$

If $p > 1$ then $\sum_{n=1}^\infty k_n r^n \sim F(1)/(1-r)^{p+1}$. If $p = 1$ then $\sum_{n=1}^\infty k_n r^n \sim (F(1) - B_p)/(1-r)^2$. In both cases we get the conclusion of the theorem by applying Theorem b to the function $\sum_{n=1}^\infty k_n r^n$.

Note 1. Theorem 5 still holds if $F \in \mathcal{C}$ [2]. But the proof of Theorem 5 does not apply if $p = 0$, $B_p \neq 0$; therefore the question whether or not Theorem 5 holds in this case is open.

Note 2. Theorems similar to those given for the class A_p could be obtained for functions $F(z) = z + \sum a_n z^n$ where the $\{a_n\}$ are not neces-

sarily real ($a_n = \alpha_n + i\beta_n$). One has to make the necessary assumptions on

$$\sin t \cdot I(\sum \alpha_n z^n), \quad \sin t \cdot I(\sum \beta_n z^n).$$

BIBLIOGRAPHY

1. S. Bochner and K. Chandrasekharan, *Fourier transforms*, Princeton Univ. Press, Princeton, N. J., 1949.
2. S. Mandelbrojt, *Quelques remarques sur les fonctions univalentes*, Bull. Sci. Math. 58 (1934), 185–200.
3. W. W. Rogosinski, *Über positive harmonische Entwicklungen und typisch reelle Potenzreihen*, Math. Z. 35 (1932), 93–121.
4. E. C. Titchmarsh, *Theory of functions*, 2nd ed., Oxford Univ. Press, Oxford.

UNIVERSITY OF WISCONSIN—MILWAUKEE