ON A CLASS OF HOLOMORPHIC FUNCTIONS[1]

NICOLAS ARTÉMIADIS

I. **Introduction.** In Part I of this paper, some inequalities, concerning functions considered in Theorem 2, are obtained. In Part II we introduce the class A_p. A function $F(z)=z+\sum a_n z^n$, $(z=re^{it})$, belongs to A_p if it is holomorphic in $|z|<1$, if the $\{a_n\}$ is real and if there exist a non-negative integer p and real numbers r_p, B_p such that:

$$\inf_{t,r}\{\sin t\cdot I[F(z)/(1-z)^p]\} = B_p \quad (t = \text{real}, 0 \leqq r_p \leqq r < 1).$$

For $p=r_p=B_p=0$ we get the class $\mathcal{C}$ of typically real functions [**3**].

Theorems concerning the coefficients $\{a_n\}$ of $F(z)$, tauberian theorems and summability methods for $\sum a_n$ are obtained.

We denote by $T(f)=\phi(t)=\int_{-\infty}^{\infty} f(x)e^{itx}dx$, the Fourier transform of $f\in L_1$.

THEOREM a [**1**, p. 20]. *If $f\in L_1$, $|f(x)|\leqq M$ in $-h\leqq x\leqq h$, $h>0$, and $\phi(t)\geqq 0$, then $\phi\in L_1$.*

The above theorem can be easily generalized as follows:

THEOREM 1. *If $f\in L_1$, $|f(x)|\leqq M$ in $-h\leqq x\leqq h$, $h>0$ and if $\alpha\leqq\arg\phi(t)\leqq\alpha+(\pi/2)$, then $\phi\in L_1$.*

PROOF. We may assume $\alpha=0$. If $\alpha\neq 0$ we consider the function $f_\alpha(x)=f(x)e^{-i\alpha}$ for which $0\leqq\arg T(f_\alpha)\leqq\pi/2$. Next put

$$F(x) = [f(x)+\overline{f(-x)}]/2, \qquad G(x) = [f(x)-\overline{f(-x)}]/2i.$$

We have

$$T(F) = R_eT(f) \geqq 0, \qquad T(G) = IT(f) \geqq 0.$$

It follows from Theorem a that $R_eT(f)\in L_1$, $IT(f)\in L_1$. Therefore $\phi\in L_1$.

Notice that since $R_eT(f)$, $IT(f)$ both belong to L_1, the inversion holds, so that

$$f(x)+\overline{f(-x)} = (1/\pi)\int_{-\infty}^{\infty} R_eT(f)e^{-itx}\,dt \qquad \text{a.e.},$$

Presented to the Society, August 27, 1964; received by the editors April 8, 1964 and, in revised form, July 1, 1964.

[1] Part of this research has been done under a grant of the University of Wisconsin Alumni Research Foundation during the summer of 1963.

$$f(x) - \overline{f(-x)} = (i/\pi) \int_{-\infty}^{\infty} IT(f) e^{-itx}\, dt \qquad \text{a.e.}$$

THEOREM 2. *Hypothesis.* $f \in L_1$; $f(x) = 0$ *for* $x < 0$; *put* $\psi(t) = \int_0^\infty f(x) \sin tx\, dx$, $\sup_t R_e[t\psi(t)] = A$, $\inf_t R_e[t\psi(t)] = B$ *and suppose* A, B *finite.*

Conclusion. For $x \geqq 0$

(a) *If* $A = 0$ *then* $2R_e\int_0^\infty f(y)dy \leqq R_e\int_0^x f(y)dy \leqq 0$.

(b) *If* $B = 0$ *then* $0 \leqq R_e^x\int_0 f(y)dy \leqq 2R_e\int_0^\infty f(y)dy$.

(c) *If* $A \neq 0$ *then* $R_e\int_0^x f(y)dy \leqq Ax$.

(d) *If* $B \neq 0$ *then* $R_e\int_0^x f(y)dy \geqq Bx$.

PROOF. Consider the functions

$$f_1(x) = (1/2i)[f(x) - f(-x)],$$

$$g(x) = \begin{matrix} e^{-\lambda x} & \text{for } x \geqq 0 \\ 0 & \text{for } x < 0 \end{matrix} \qquad \lambda > 0,$$

$$g_1(x) = (1/2i)[g(x) - g(-x)].$$

We find:

$$T(g_1) = t/(t^2 + \lambda^2), \quad T(f_1) = \psi(t), \quad T[Ae^{-\lambda|x|}/2\lambda] = A/(\lambda^2 + t^2).$$

Also

$$\begin{aligned} R_eT[(Ae^{-\lambda|x|}/2\lambda) - f_1 * g_1] &= R_e[(A - t\psi(t))/(t^2 + \lambda^2)] \\ &= [A - R_et\psi(t)]/(t^2 + \lambda^2) > 0, \end{aligned} \tag{2.1}$$

$$\begin{aligned} R_eT[f_1 * g_1 - (Be^{-\lambda|x|}/2\lambda)] &= R_e[(t\psi(t) - B)/(t^2 + \lambda^2)] \\ &= [R_et\psi(t) - B]/(t^2 + \lambda^2) \geqq 0. \end{aligned} \tag{2.2}$$

We have:

$$f_1 * g_1 = (1/4) \int_0^\infty [f(x+y) - f(-y-x) - f(x-y) + f(y-x)] e^{-\lambda y}\, dy. \tag{2.3}$$

It follows from (2.1), (2.2) that the functions:

$$(Ae^{-\lambda|x|}/2\lambda) - R_e[f_1 * g_1], \qquad R_e[f_1 * g_1] - (Be^{-\lambda|x|}/2\lambda)$$

are bounded, continuous and they have non-negative Fourier transforms, which, by Theorem a, belong to L_1. Therefore the inversion formula holds everywhere:

$$\frac{Ae^{-\lambda|x|}}{2\lambda} - R_e[f_1 * g_1] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A - R_et\psi(t)}{t^2 + \lambda^2} e^{-ixt}\, dt \quad \text{everywhere}$$

$$R_e[f_1 * g_1] - \frac{Be^{-\lambda|x|}}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{R_e t\psi(t) - B}{t^2 + \lambda^2} e^{-ixt}\, dt \quad \text{everywhere.}$$

By taking the absolute values of both members we get:

$$\begin{aligned} \left| (Ae^{-\lambda|x|}/2\lambda) - R_e[f_1 * g_1] \right| &\leqq \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{A - R_e t\psi(t)}{t^2 + \lambda^2}\, dt \\ &= (A/2\lambda) - R_e[f_1 * g_1]_{x=0}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} \left| R_e[f_1 * g_1] - (Be^{-\lambda|x|}/2\lambda) \right| &\leqq \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{R_e t\psi(t) - B}{t^2 + \lambda^2}\, dt \\ &= R_e[f_1 * g_1]_{x=0} - (B/2\lambda). \end{aligned} \tag{2.5}$$

For $x = 0$, we get from (2.3):

$$R_e[f_1 * g_1]_{x=0} = (1/2)R_e \int_0^{\infty} f(y)e^{-\lambda y}\, dy.$$

Suppose $A = 0$. Then (2.4) becomes:

$$\left| R_e[f_1 * g_1] \right| \leqq - R_e[f_1 * g_1]_{x=0}$$

or

$$\left| R_e(1/4) \int_0^{\infty} [f(x + y) - f(-y - x) - f(x - y) + f(y - x)]e^{-\lambda y}\, dy \right|$$
$$\leqq - (1/2)R_e \int_0^{\infty} f(y)e^{-\lambda y}\, dy.$$

The conclusion (a) follows if in the last inequality we let $\lambda \to 0+$. If $B = 0$, then conclusion (b) follows from (2.5) in the same way. Suppose now $A \neq 0$. We get from (2.4):

$$\frac{Ae^{-\lambda|x|} - A}{2\lambda} - R_e \frac{1}{4}\int_0^{\infty} [f(x+y) - f(-y-x) - f(x-y) + f(y-x)]e^{-\lambda y}\, dy$$
$$\leqq -R_e \frac{1}{2}\int_0^{\infty} f(y)e^{-\lambda y}\, dy$$

and the conclusion (c) follows if $\lambda \to 0+$. If $B \neq 0$, conclusion (d) follows from (2.5) in a similar way.

COROLLARY. *Hypothesis.* $f \in L_1$; $f(x) = 0$ *for* $x < 0$; *put* $\psi(t) = \int_0^{\infty} f(x) \sin tx\, dx$, $\sup_t I[t\psi(t)] = A^*$, $\inf_t I[t\psi(t)] = B^*$, *and suppose* A^*, B^* *finite.*

Conclusion. The conclusion of Theorem 2 holds if we replace A, B, R_e *by* A^*, B^*, I *respectively.*

PROOF. Put $\psi^*(t) = \int_0^\infty (-i) f(x) \sin tx\, dx$. We have $I[t\psi(t)] = R_e[t\psi^*(t)]$ and the corollary follows if in the conclusion of Theorem 2 we replace f by $-if$.

II. **Definition of the class $\mathcal{C}$ of typically real functions [3].** A function $F(z) = z + \sum_{n=2}^\infty a_n z^n$ $(z = re^{it})$ is said to be typically real in the circle $|z| < 1$, if it is holomorphic in this circle and if $F(z)$ is real for real values of z, but for no other values in $|z| < 1$.

It follows from the above definition that the coefficients $\{a_n\}$ are real. Also, one can prove, that $F \in \mathcal{C}$ if and only if, sign $IF(z) =$ sign Iz. This last relation is equivalent to: $\sin t \cdot IF(re^{it}) \geqq 0$.

As we mentioned in the introduction of this paper, the class A_p is a generalization of the class $\mathcal{C}$. More precisely, we notice that $\mathcal{C}$ is a proper subclass of A_0. In fact, $F(z) = [z/(1-z)] + 2z^2$ belongs to A_0 but not to $\mathcal{C}$.

Also, if $F \in A_p$, we have for $r < 1$ and $t = 0$, $\sin t \cdot I[F(z)/(1-z)^p] = 0$; therefore $B_p \leqq 0$.

THEOREM 3. *If $F \in A_0$, then*:
(a) $|a_{n+1} - a_{n-1}| \leqq 2 - 4B_0$,
(b) $|a_n| \leqq n(1 - 2B_0)$,
(c) $1 + a_2 + \cdots + a_{n-1} + (a_n/2) \geqq B_0 n$
where $n = 1, 2, 3, \cdots, a_0 = 0$.

PROOF. Put:

$$f(x) = \begin{cases} a_n r^n + a_{n+1} r^{n+1} & \text{for } n \leqq x < n+1, \\ 0 & \text{for } x < 0, \end{cases}$$

$$n = 0, 1, 2, \cdots, 0 \leqq r_0 \leqq r < 1.$$

$$\psi(t) = \int_0^\infty f(x) \sin tx\, dx.$$

We find:

$$\begin{aligned} t\psi(t) &= 2 \sin t \cdot IF(z) \\ &= [r + a_2 r^2 \cos t + (a_3 r^3 - r) \cos 2t + (a_4 r^4 - a_2 r^2) \cos 3t + \cdots] \\ &\geqq 2B_0. \end{aligned}$$

Multiplying both sides by $1 \pm \cos nt$ and integrating from 0 to 2π we get the inequality:

$$\pi r[2 \pm (a_{n+1} - a_{n-1} r^{-2}) r^{n-1}] \geqq 2B_0 \cdot 2\pi$$

or $|a_{n+1} - a_{n-1}| \leqq 2 - 4B_0$, $(n \geqq 1)$. Next put $B_r = \inf_t [t\psi(t)]$. We have

$B_r \geqq 2B_0$. The function f satisfies the assumptions of Theorem 2, therefore $r+a_2r^2+\cdots+a_{n-1}r^{n-1}+(a_nr^n/2) \geqq B_rn/2 \geqq B_0n$. The conclusion (c) follows if we let $r\to 1-$.

Note. For $r_0=B_0=0$ we get the well-known inequalities: $|a_n| \leqq n$, $1+a_2+\cdots+a_{n-1}+(a_n/2)\geqq 0$ for the functions of the class $\mathfrak{C}$ ([2], [3]).

An analogous theorem can be given for functions of the class A_p ($p\geqq 1$).

Put $s_n^{(0)}=a_n$, $s_n^{(p)}=\sum_{i=1}^n s_i^{(p-1)}$ ($p=1, 2, \cdots$). It is easy to see that if $F\in A_p$ then $F_p(z)=\sum_{n=1}^\infty s_n^{(p)}z^n$ belongs to A_0. Applying Theorem 3 to F_p we get:

THEOREM 3*. *If* $F\in A_p$ *then*

(a) $|s_{n+1}^{(p)} - s_{n-1}^{(p)}| \leqq 2 - 4B_p$,

(b) $|s_n^{(p)}| \leqq n(1-2B_p)$ $(n\geqq 1,\ s_0^{(p)}=0)$,

(c) $1+s_2^{(p)}+s_3^{(p)}+\cdots+s_{n-1}^{(p)}+(s_n^{(p)}/2)\geqq B_pn$.

THEOREM 4. *If* $F\in A_0$ *and* $F(r)\sim(1-r)^{-1}$ *then*

$$\sum_{k=1}^n (s_k^{(1)}/k) \sim n \quad (n\to\infty).$$

PROOF. By (c) of Theorem 3 we have: $(s_n^{(1)}/n)-(a_n/2n)-B_0\geqq 0$. Also

$$\sum_{n=1}^\infty [(s_n^{(1)}/n) - (a_n/2n) - B_0]r^n \sim [(1-B_0)/(1-r)].$$

It follows from Hardy-Littlewood's theorem [4, p. 226]:

$$\sum_{k=1}^n [(s_k^{(1)}/k) - (a_k/2k) - B_0] \sim n(1-B_0) \qquad (n\to\infty)$$

or

$$(*) \qquad \sum_{k=1}^n [(s_k^{(1)}/k) - (a_k/2k)] \sim n \qquad (n\to\infty).$$

By (a) of Theorem 3 we have:

$$2-4B_0+a_{n+1}-a_{n-1}\geqq 0,$$

$$\sum_{n=1}^\infty (2-4B_0+a_{n+1}-a_{n-1})r^n \sim [(2-4B_0)/(1-r)].$$

Applying again Hardy-Littlewood's theorem we get:

$$\lim_{n \to +\infty} [(a_n + a_{n+1})/n] = 0.$$

We write:

$$\frac{a_{n+1} + a_{n+2}}{n} = \frac{n+1}{n}\left(\frac{a_{n+1}}{n+1} + \frac{a_{n+2}}{n+2}\right) + \frac{a_{n+2}}{n(n+2)}.$$

Since $|a_n| \leqq n(1-2B_0)$ we have $\lim_{n\to\infty}[a_{n+2}/n(n+2)] = 0$. Therefore $\lim_{n\to\infty}[(a_{n+1}+a_{n+2})/n] = 0 = \lim_{n\to\infty}[(a_{n+1}/n+1)+(a_{n+2}/n+2)]$,

$$1 + (a_2/2) + (a_3/3) + \cdots (a_n/n) = 0(n) \qquad (n \to +\infty),$$

and the conclusion of the theorem follows from (*).

To generalize Theorem 4 we state the following:

THEOREM b [4, Ex. 8, p. 242]. *If* $a_n \geqq 0$ *and* $(\sum a_n x^n) \sim (1-x)^{-\alpha}$, $(\alpha > 1)$ *then* $\sum_{k=1}^{n} a_k \sim (n^\alpha/\Gamma(\alpha+1))$, $(n \to \infty)$.

THEOREM 4*. *If* $F \in A_p$ *and* $F(r) \sim (1-r)^{-\alpha}$, $(\alpha > 1)$ *then* $\sum_{k=1}^{n} (s_k^{p+1)}/k) \sim (n^{\alpha+p}/\Gamma(\alpha+p+1))$, $(n \to \infty)$.

PROOF. Put $F_p(x) = \sum_{n=1}^{\infty} s_n^{(p)} z^n$. Notice that $F_p \in A_0$, $F_p(r) \sim (1-r)^{-(\alpha+p)}$ and use Theorems 3 and b. The proof is very similar to the proof of Theorem 4.

The following theorem provides a summability method for $\sum a_n$, where $F(z) = \sum a_n z^n$ belongs to A_p. A similar theorem is given in [2] for the class $\mathfrak{C}$.

THEOREM 5. *If* $F \in A_p$, $(p \geqq 1)$ *and* $\lim_{r\to 1-} F(r) = F(1)$ *exists and is positive then*:

$$\lim_{n\to\infty} \frac{\Gamma(p+1)}{n^{p+1}} [1 + s_2^{(p+1)} + \cdots + s_{n-1}^{(p+1)} + (s_n^{(p+1)}/2)] = F(1).$$

PROOF. From (c) of Theorem 3* it follows that:

$$k_n = s_n^{(p+1)} - (s_n^{(p)}/2) - B_p n \geqq 0.$$

If $p > 1$ then $\sum_{n=1}^{\infty} k_n r^n \sim F(1)/(1-r)^{p+1}$. If $p=1$ then $\sum_{n=1}^{\infty} k_n r^n \sim (F(1) - B_p)/(1-r)^2$. In both cases we get the conclusion of the theorem by applying Theorem b to the function $\sum_{n=1}^{\infty} k_n r^n$.

Note 1. Theorem 5 still holds if $F \in \mathfrak{C}$ [2]. But the proof of Theorem 5 does not apply if $p=0$, $B_p \neq 0$; therefore the question whether or not Theorem 5 holds in this case is open.

Note 2. Theorems similar to those given for the class A_p could be obtained for functions $F(z) = z + \sum a_n z^n$ where the $\{a_n\}$ are not necessarily real ($a_n = \alpha_n + i\beta_n$). One has to make the necessary assumptions on

$$\sin t \cdot I\left(\sum \alpha_n z^n\right), \qquad \sin t \cdot I\left(\sum \beta_n z^n\right).$$

Bibliography

1. S. Bochner and K. Chandrasekharan, *Fourier transforms*, Princeton Univ. Press, Princeton, N. J., 1949.

2. S. Mandelbrojt, *Quelques remarques sur les fonctions univalentes*, Bull. Sci. Math. **58** (1934), 185–200.

3. W. W. Rogosinski, *Über positive harmonische Entwicklungen und typisch reele Potenzreihen*, Math. Z. **35** (1932), 93–121.

4. E. C. Titchmarch, *Theory of functions*, 2nd ed., Oxford Univ. Press, Oxford.

University of Wisconsin—Milwaukee