THE SHILOV BOUNDARY OF THE ALGEBRA OF MEASURES ON A GROUP[1]

JOSEPH L. TAYLOR

If G is a locally compact abelian topological group and $M(G)$ denotes the algebra of bounded regular Borel measures on G under convolution multiplication, then $M(G)$ is a convolution measure algebra in the sense of [1]. In [1] we showed that the maximal ideal space of any such algebra $\mathfrak{M}$ can be represented as the semigroup $\hat{S}$ of all semicharacters on some compact abelian topological semigroup S. S is called the structure semigroup of the algebra $\mathfrak{M}$. If $H = \{h \in \hat{S}: |h(s)| = 0 \text{ or } 1 \text{ for } s \in S\}$, then the Gelfand transform $\hat{\mu}$ of each element μ of $\mathfrak{M}$ attains its maximum modulus on H (cf. [1, Theorem 3.3]). Hence the closure $\overline{H}$ of H in the Gelfand topology contains the Shilov boundary of $\mathfrak{M}$. In [1] we show that when $\mathfrak{M} = M(G)$ for some nondiscrete locally compact topological group G, then H is a proper subset of $\hat{S}$. In this paper we show that there is at least one group G for which $\overline{H}$ is a proper subset of $\hat{S}$. Hence, for this group G, the Shilov boundary of $M(G)$ is a proper subset of the maximal ideal space of $M(G)$.

For each positive integer n let T_n be the multiplicative two point group $\{1, -1\}$ and set $G = \prod_{n=1}^{\infty} T_n$. G is a compact abelian topological group. For each n we let χ_n be the function which projects G onto its nth coordinate. Each χ_n is a character in the dual group $\hat{G}$ of G and each $k \in \hat{G}$ is either the identity or a finite product of distinct χ_n's.

S will denote the structure semigroup of $M(G)$ and $\hat{S}$ the semigroup of all continuous semicharacters on S. $\mu \to \mu_S$ is the natural imbedding of $M(G)$ into $M(S)$ (cf. [1, Theorem 2.3]). The Gelfand transform $\hat{\mu}$ of $\mu \in M(G)$ is described by the equation $\hat{\mu}(f) = \int f \, d\mu_S$ for $f \in \hat{S}$.

We are interested in a particular class of measures μ in $M(G)$. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of numbers in $[0, 1)$ and for each n let μ_n be the measure on T_n defined by $\mu_n(1) = 2^{-1}(1 + r_n)$ and $\mu_n(-1) = 2^{-1}(1 - r_n)$. Each μ_n is a strictly positive measure of norm one on T_n. Let μ be the measure in $M(G)$ which is the infinite product of the μ_n. That is, if U is any neighborhood in G of the form $U = \{g \in G: \chi_n(g)$

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$=e_n$, $n=1, 2, \cdots, m\}$, where $\{e_n\}_{n=1}^m$ is any n-tuple of 1's and -1's, then $\mu(U)=\prod_{n=1}^m 2^{-1}(1+e_n r_n)$. Note that if k is any character of the form $k=\prod_{i=1}^m \chi_{n_i}$, where $\chi_{n_i}\neq\chi_{n_j}$ for $i\neq j$, then

$$\int k\,d\mu = \prod_{i=1}^m [2^{-1}(1+r_{n_i}) - 2^{-1}(1-r_{n_i})] = \prod_{i=1}^m r_{n_i}.$$

We denote by $\mathfrak{L}(\mu)$ the Banach space of all measures in $M(G)$ which are absolutely continuous with respect to μ. The adjoint space of $\mathfrak{L}(\mu)$ is $L_\infty(\mu)$. Hence each function $f\in\hat{S}$ determines a function f' in $L_\infty(\mu)$, such that $\int f\,d\nu_S=\int f'\,d\nu$ for each $\nu\in\mathfrak{L}(\mu)$. The map $\nu\to\nu_S$ is an L-homomorphism of $\mathfrak{L}(\mu)$ into $M(S)$ (cf. [**1**, Definition 1.3 and Theorem 2.3]) and its adjoint map is the map $f\to f'$. Thus, by Theorem 1.2 of [**1**], $f\to f'$ preserves pointwise multiplication and is a homomorphism of the semigroup $\hat{S}$. We are interested in characterizing the image of $\hat{S}$ in $L_\infty(\mu)$.

LEMMA 1. *Each function f' for $f\in\hat{S}$ is of the form* $\lim_m a\prod_{n=1}^m \chi_n^{e_n}$, *where a is a constant with $|a|\leqq 1$, $e_n=0$ or 1 for $n=1, 2, \cdots$, and the limit is in $L_1(\mu)$ norm.*

PROOF. If $g\in G$ we denote by δ_g the point measure at g. The function $f\in\hat{S}$ defines a multiplicative function k (not necessarily continuous) on G by $k(g)=\int f\,d(\delta_g)_S=\hat{\delta}_g(f)$. If G_0 denotes the subgroup of G consisting of all g for which $\{\chi_n(g)\}_{n=1}^\infty$ is eventually 1, then there exists a sequence $\{e_n\}_{n=1}^\infty$ of 0's and 1's, such that $k(g)=\prod_{n=1}^\infty \chi_n^{e_n}(g)$ for $g\in G_0$.

Now for each positive integer m set $U_m=\{g\in G\colon \chi_n(g)=1$ if $n\leqq m\}$ and $E_m=\{g\in G\colon \chi_n(g)=1$ if $n>m\}$. E_m contains 2^m elements, U_m is a compact neighborhood of the identity, and $\{gU_m\colon g\in E_m\}$ is a pairwise disjoint cover of G. Let π_m be the characteristic function of U_m. For each m we choose a collection of numbers $\{b_{m,g}\}_{g\in E_m}$ with $|b_{m,g}|\leqq 1$, which minimizes the number

$$\int\left| f'(g) - \sum_{g'\in E_m} b_{m,g'}\pi_m(gg')\right| d\mu(g).$$

We set $h_m(g)=\sum_{g'\in E_m} b_{m,g'}\pi_m(gg')$. The sequence $\{\int|h_m-f'|\,d\mu\}_{m=1}^\infty$ is nonincreasing and, since the continuous simple step functions of norm $\leqq 1$ are dense in the unit ball of $L_1(\mu)$, it follows that this sequence converges to zero.

Fix m and for $g\in E_m$ and V a Borel set of G define $\nu_g(V)=\mu(V\cap gU_m)$, then

$$\nu_{g_1} = \|\nu_{g_2}\|^{-1}\|\nu_{g_1}\|\delta_{g_1 g_2^{-1}} \cdot \nu_{g_2}$$
$$= \prod_{\chi_n(g_1)=-1} (1+r_n)^{-1}(1-r_n) \prod_{x_n(g_2)=-1} (1-r_n)^{-1}(1+r_n)\delta_{g_1 g_2^{-1}} \cdot \nu_{g_2}.$$

Also, it follows from the definitions of f' and k that $f'(gg_1) = k(g_1)f'(g) = \prod_{n=1}^{m} \chi_n^{e_n}(g_1)f'(g)$ a.e./μ for each $g_1 \in E_m$. Choose $g_0 \in E_m$ such that $\|\nu_{g_0}\|^{-1}\int|f' - h_m|\,d\nu_{g_0} = \min_{g\in E_m} \|\nu_g\|^{-1}\int|f' - h_m|\,d\nu_g$ and let $a_m = b_{m,g_0}k(g_0)$ and $k_m = \prod_{n=1}^{m} \chi_n^{e_n}$. Then

$$\int |f' - a_m k_m|\,d\mu = \sum_{g_1\in E_m} \int |f' - a_m k_m|\,d\nu_{g_1}$$
$$= \sum_{g_1\in E_m} \|\nu_{g_0}\|^{-1}\|\nu_{g_1}\| \int |f' - a_m k_m|\,d(\delta_{g_1 g_0^{-1}} \cdot \nu_{g_0})$$
$$= \sum_{g_1\in E_m} \|\nu_{g_0}\|^{-1}\|\nu_{g_1}\| \int |f'(g_1 g_0^{-1} g) - a_m k_m(g_1 g_0^{-1} g)|\,d\nu_{g_0}(g)$$
$$= \sum_{g_1\in E_m} \|\nu_{g_0}\|^{-1}\|\nu_{g_1}\| \int |k_m(g_1 g_0^{-1})(f'(g) - a_m k_m(g))|\,d\nu_{g_0}(g)$$
$$= \sum_{g_1\in E_m} \|\nu_{g_0}\|^{-1}\|\nu_{g_1}\| \int |f' - h_m|\,d\nu_{g_0}$$
$$\leqq \sum_{g_1\in E_m} \int |f' - h_m|\,d\nu_{g_1} = \int |f' - h_m|\,d\mu.$$

That is, $\int|f' - a_m k_m|\,d\mu \leqq \int|f' - h_m|\,d\mu$. Hence

$$\{a_m k_m\}_{m=1}^{\infty} = \left\{a_m \prod_{n=1}^{m} \chi_n^{e_n}\right\}_{m=1}^{\infty}$$

converges in $L_1(\mu)$ norm to f'. If a is a cluster point of the sequence $\{a_m\}_m$, then $\{a\prod_{n=1}^{m} \chi_n^{e_n}\}_{m=1}^{\infty}$, also converges to f' in $L_1(\mu)$ norm. This completes the proof.

LEMMA 2. *If* $\limsup_n r_n < 1$ *and* $f' = a \lim_m \prod_{n=1}^{m} \chi_n^{e_n}$ *as in Lemma* 1, *with* $|a| > 0$, *then there exists* M, *such that* $e_n = 0$ *if* $n > M$. *Hence* $f' = ak$ *where* $k = \prod_{n=1}^{M} \chi_n^{e_n} \in \hat{G}$.

PROOF.

$$\int \left| \prod_{n=1}^{m-1} \chi_n^{e_n} - \prod_{n=1}^{m} \chi_n^{e_n} \right| d\mu = \int |1 - \chi_m^{e_m}|\,d\mu = e_m(1 - r_m).$$

Hence if $\{\prod_{n=1}^{M} \chi_n^{e_n}\}_{m=1}^{\infty}$ converges in $L_1(\mu)$ norm, then either $\limsup_n r_n = 1$ or $\{e_n\}_{n=1}^{\infty}$ is eventually zero.

For each positive integer n let A_n be the subset of $[0, 1]$ consisting of 1 and all finite products $\prod_{i=1}^{m} r_{n_i}$ with $n < n_i$ for $i = 1, 2, \cdots, m$ and $n_i \neq n_j$ if $i \neq j$.

LEMMA 3. *If* $\limsup_n r_n < 1$, *then the closure of* $\hat{G}$ *in the weak-* topology of* $L_\infty(\mu)$ *is* $\{ak : k \in \hat{G} \text{ and } a \in \bigcap_n \bar{A}_n\}$.

PROOF. If $a \in \bigcap_n \bar{A}_n$ then there is a sequence $\{\{p_{i,n}\}_{i=1}^{m_n}\}_{n=1}^{\infty}$ of tuples of distinct integers, with $p_{i,n} \geqq n$, such that $\lim_n \prod_{i=1}^{m_n} r_{p_{i,n}} = a$. If $k \in \hat{G}$ then k is a product of χ_p's with $p \leqq M$ for some integer M; set $h_n = k \prod_{i=1}^{m_n} \chi_{p_{i,n}} \in \hat{G}$. If U is any open-compact rectangle in G of the form $U = \{g \in G : \chi_{q_j}(g) = \sigma_j\}$ for $j = 1, 2, \cdots, u$ where $\{\sigma_j\}_{j=1}^{u}$ is any u-tuple of 1's and -1's, then $\int_U h_n \, d\mu = \prod_{i=1}^{m_n} r_{p_{i,n}} \int_U k \, d\mu$ provided $n > q_j$ for $j = 1, 2, \cdots, u$, and $n > M$. Hence $\lim_n \int_U h_n \, d\mu = \int_U ak \, d\mu$. From this fact and the fact that $\{h_n\}_{n=1}^{\infty}$ is uniformly bounded it follows that $\lim_n h_n = ak$ in the weak-* topology of $L_\infty(\mu)$.

Conversely, suppose h is in the weak-* closure of $\hat{G}$ in $L_\infty(\mu)$. Then $h = f'$ for some $f \in \hat{S}$ and hence, by Lemma 2, $h = ak$ for some a with $|a| \leqq 1$ and $k \in \hat{G}$. Let $\{k_\alpha\}$ be a net in $\hat{G}$ converging weak-* to ak. Then $\lim_\alpha kk_\alpha = a$. If a is not 1 then we may assume that $kk_\alpha = \prod_{i=1}^{m_\alpha} \chi_{n_{i,\alpha}}$, where $n_{i,\alpha} \neq n_{j,\alpha}$ if $i \neq j$. Then $\lim \int kk_\alpha \, d\mu = \lim \prod_{i=1}^{m_\alpha} r_{n_{i,\alpha}} = a$. Also, since the weak-* limit of $\{kk_\alpha\}$ is a constant, it follows that, given n, eventually $n_{i,\alpha} \geqq n$ for $i = 1, 2, \cdots, m_\alpha$. Hence $a \in \bigcap_m \bar{A}_m$. This completes the proof.

THEOREM 1. *If* $\limsup_n r_n < 1$, $k \in \hat{G}$, *and* $0 < |a| < 1$, *then* $ak = f'$ *for some* f *in the Shilov boundary of* $M(G)$ *if and only if* $|a| \in \bigcap_n \bar{A}_n$.

PROOF. If $|a| \in \bigcap_n \bar{A}_n$ then, by Lemma 3, $|a|$ is in the weak-* closure in $L_\infty(\mu)$ of $\hat{G}$. It follows that there exists h in the closure of $\hat{G}$ in $\hat{S}$ such that $h' = |a|$, that is, h is identically $|a|$ on the carrier of μ_S in S. Then h is identically $|a|^n$ on the carrier of μ_S^n in S for each n. Since $0 < |a| < 1$ it follows that carrier $(\mu_S^n) \cap$ carrier $(\mu_S^m) = \emptyset$ for $n \neq m$. Let $\nu(V) = \int_V |a|^{-1} \bar{a} \bar{k} \, d\mu$ for each Borel set V of G. Then carrier $(\nu_S^n) \cap$ carrier $(\nu_S^m) = \emptyset$ for $n \neq m$, and hence

$$\|(\nu + \delta_e)^n\| = \left\| \sum_{m=0}^{n} \binom{n}{m} \nu^m \right\| = \sum_{m=0}^{n} \binom{n}{m} \|\nu^m\| = \sum_{m=0}^{n} \binom{n}{m} = 2^n,$$

where e is the identity of G and δ_e is the point measure at e. Thus $\nu + \delta_e$ has spectral radius 2 and it follows that there exists h_1 in the Shilov boundary of $M(G)$, such that $|(\hat{\nu} + \hat{\delta}_e)(h_1)| = |\hat{\nu}(h_1) + 1| = 2$. Since $\|\nu\| = 1$, $\hat{\nu}(h_1)$ must be 1. Then $\int |a|^{-1} \bar{a} \bar{k} h_1' \, d\mu = 1$ and we conclude that $h_1' = |a|^{-1} ak$ and $(hh_1)' = ak$. Now the Shilov boundary is clearly invariant under multiplication by elements of $\hat{G}$ and, since the

Shilov boundary is closed, it is invariant under multiplication by elements of the closure of $\hat{G}$ in $\hat{S}$. Hence hh_1 is in the Shilov boundary.

Conversely, suppose $ak=f'$ where f is in the Shilov boundary. By Theorem 3.3 of [1], f is the limit of a net $\{h_\alpha\} \subset H = \{h \in \hat{S}: |h(s)| = 0$ or 1 for $s \in S\}$. By Lemmas 1 and 2, there exist numbers a_α, $|a_\alpha| = 0$ or 1, and characters k_α, such that $h'_\alpha = a_\alpha k_\alpha$ for each α. Clearly, $\lim_\alpha a_\alpha = a/|a|$ and $\lim_\alpha k_\alpha = |a| k$ in the weak-* topology of $L_\infty(\mu)$. Hence, by Lemma 3, $|a| \in \bigcap_n \overline{A}_n$.

THEOREM 2. *The Shilov boundary of $M(G)$ is a proper subset of $\hat{S}$.*

PROOF. If $\{r_n\}_{n=1}$ is chosen such that $0 < \limsup_n r_n < 1$, then there is a positive number $a \in \bigcap_n \overline{A}_n$. Then $a = f'_a$ for some $f_a \in \hat{S}$, by Theorem 1, where f_a may be chosen such that $f_a(s) \geqq 0$ for each $s \in S$. Hence, $f_a^z \in \hat{S}$ for each complex number z with $\operatorname{Re} z > 0$, and $f_a^{z\prime} = a^z$. It follows that for each b in the unit disc there exists $f_b \in \hat{S}$, such that $f'_b = b$. By Theorem 1, f_b may be chosen from the Shilov boundary if and only if $|b| \in \bigcap \overline{A}_n$. However $\bigcap_n \overline{A}_n \subset [0, \limsup_n r_n] \cup 1$ which is a proper subset of $[0, 1]$. This completes the proof.

REFERENCES

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HARVARD UNIVERSITY