THE SHILOV BOUNDARY OF THE ALGEBRA
OF MEASURES ON A GROUP\(^1\)

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If \( G \) is a locally compact abelian topological group and \( M(G) \) denotes the algebra of bounded regular Borel measures on \( G \) under convolution multiplication, then \( M(G) \) is a convolution measure algebra in the sense of [1]. In [1] we showed that the maximal ideal space of any such algebra \( \mathcal{M} \) can be represented as the semigroup \( \mathcal{S} \) of all semicharacters on some compact abelian topological semigroup \( \mathcal{S} \). \( \mathcal{S} \) is called the structure semigroup of the algebra \( \mathcal{M} \). If \( H = \{ h \in \mathcal{S} : |h(s)| = 0 \text{ or } 1 \text{ for } s \in \mathcal{S} \} \), then the Gelfand transform \( \hat{\mu} \) of each element \( \mu \) of \( \mathcal{M} \) attains its maximum modulus on \( H \) (cf. [1, Theorem 3.3]). Hence the closure \( \overline{H} \) of \( H \) in the Gelfand topology contains the Shilov boundary of \( \mathcal{M} \). In [1] we show that when \( \mathcal{M} = M(G) \) for some nondiscrete locally compact topological group \( G \), then \( H \) is a proper subset of \( \mathcal{S} \). In this paper we show that there is at least one group \( G \) for which \( \mathcal{H} \) is a proper subset of \( \mathcal{S} \). Hence, for this group \( G \), the Shilov boundary of \( M(G) \) is a proper subset of the maximal ideal space of \( M(G) \).

For each positive integer \( n \) let \( T_n \) be the multiplicative two point group \( \{1, -1\} \) and set \( G = \prod_{n=1}^{\infty} T_n \). \( G \) is a compact abelian topological group. For each \( n \) we let \( \chi_n \) be the function which projects \( G \) onto its \( n \)th coordinate. Each \( \chi_n \) is a character in the dual group \( \hat{G} \) of \( G \) and each \( k \in \hat{G} \) is either the identity or a finite product of distinct \( \chi_n \)’s.

\( \mathcal{S} \) will denote the structure semigroup of \( M(G) \) and \( \mathcal{S} \) the semigroup of all continuous semicharacters on \( \mathcal{S} \). \( \mu \rightarrow \mu_S \) is the natural imbedding of \( M(G) \) into \( M(\mathcal{S}) \) (cf. [1, Theorem 2.3]). The Gelfand transform \( \hat{\mu} \) of \( \mu \in M(G) \) is described by the equation \( \hat{\mu}(f) = \int f \, d\mu_S \) for \( f \in \mathcal{S} \).

We are interested in a particular class of measures \( \mu \) in \( M(G) \). Let \( \{r_n\}_{n=1}^{\infty} \) be a sequence of numbers in \([0, 1)\) and for each \( n \) let \( \mu_n \) be the measure on \( T_n \) defined by \( \mu_n(1) = 2^{-1}(1 + r_n) \) and \( \mu_n(-1) = 2^{-1}(1 - r_n) \). Each \( \mu_n \) is a strictly positive measure of norm one on \( T_n \). Let \( \mu \) be the measure in \( M(G) \) which is the infinite product of the \( \mu_n \). That is, if \( U \) is any neighborhood in \( G \) of the form \( U = \{g \in G : \chi_n(g) \)
\( e_n, n = 1, 2, \ldots, m \), where \( \{ e_n \}_{n=1}^{m} \) is any \( n \)-tuple of 1's and \(-1\)'s, then \( \mu(U) = \prod_{n=1}^{m} 2^{-1}(1 + e_n r_n) \). Note that if \( k \) is any character of the form \( k = \prod_{i=1}^{m} x_{\alpha_i}, \) where \( x_{\alpha_i} \neq x_{\beta_j} \) for \( i \neq j \), then

\[
\int k \, d\mu = \prod_{i=1}^{m} [2^{-1}(1 + r_n) - 2^{-1}(1 - r_n)] = \prod_{i=1}^{m} r_n.
\]

We denote by \( \mathcal{S}(\mu) \) the Banach space of all measures in \( M(G) \) which are absolutely continuous with respect to \( \mu \). The adjoint space of \( \mathcal{S}(\mu) \) is \( L^1(\mu) \). Hence each function \( f \in \mathcal{S} \) determines a function \( f^* \) in \( L^1(\mu) \), such that \( \int f \, d\nu_S = \int f^* \, d\nu \) for each \( \nu \in \mathcal{S}(\mu) \). The map \( \nu \rightarrow \nu_S \) is an \( L \)-homomorphism of \( \mathcal{S}(\mu) \) into \( M(S) \) (cf. [1, Definition 1.3 and Theorem 2.3]) and its adjoint map is the map \( f \rightarrow f^* \). Thus, by Theorem 1.2 of [1], \( f \rightarrow f^* \) preserves pointwise multiplication and is a homomorphism of the semigroup \( \mathcal{S} \). We are interested in characterizing the image of \( \mathcal{S} \) in \( L^1(\mu) \).

**Lemma 1.** Each function \( f^* \) for \( f \in \mathcal{S} \) is of the form \( \lim_m a \prod_{n=1}^{m} x_{\alpha_n} \), where \( a \) is a constant with \( |a| \leq 1 \), \( e_n = 0 \) or 1 for \( n = 1, 2, \ldots, m \), and the limit is in \( L^1(\mu) \) norm.

**Proof.** If \( g \in G \) we denote by \( \delta_g \) the point measure at \( g \). The function \( f \in \mathcal{S} \) defines a multiplicative function \( k \) (not necessarily continuous) on \( G \) by \( k(g) = \int f \, d(\delta_g)_S = \delta_g(f) \). If \( G_0 \) denotes the subgroup of \( G \) consisting of all \( g \) for which \( \{ x_n(g) \}_{n=1}^{\infty} \) is eventually 1, then there exists a sequence \( \{ e_n \}_{n=1}^{\infty}, \) of 0's and 1's, such that \( k(g) = \prod_{n=1}^{\infty} x_{\alpha_n}(g) \) for \( g \in G_0 \).

Now for each positive integer \( m \) set \( U_m = \{ g \in G : x_n(g) = 1 \text{ if } n \leq m \} \) and \( E_m = \{ g \in G : x_n(g) = 1 \text{ if } n > m \} \). \( E_m \) contains \( 2^m \) elements, \( U_m \) is a compact neighborhood of the identity, and \( \{ g U_m : g \in E_m \} \) is a pairwise disjoint cover of \( G \). Let \( \pi_m \) be the characteristic function of \( U_m \). For each \( m \) we choose a collection of numbers \( \{ b_{m, o} \}_{o \in E_m} \) with \( |b_{m, o}| \leq 1 \), which minimizes the number

\[
\int |f'(g) - \sum_{o' \in E_m} b_{m, o'} \pi_m(gg')| \, d\mu(g).
\]

We set \( h_m(g) = \sum_{o' \in E_m} b_{m, o'} \pi_m(gg') \). The sequence \( \{ f | h_m - f' | d\mu \}_{m=1}^{\infty} \) is nonincreasing and, since the continuous simple step functions of norm \( \leq 1 \) are dense in the unit ball of \( L_1(\mu) \), it follows that this sequence converges to zero.

Fix \( m \) and for \( g \in E_m \) and \( V \) a Borel set of \( G \) define \( \nu_g(V) = \mu(V \cap g U_m) \), then
\[ v_{01} = \left\| v_{21} \right\|^{-1} \left\| v_{21} \right\| \delta_{21} (g_{1})^{-1} \cdot v_{21} \]
\[ = \prod_{x_{n}(g_{1}) = 1} (1 + r_{n})^{-1} (1 - r_{n}) \prod_{x_{n}(g_{1}) = -1} (1 - r_{n})^{-1} (1 + r_{n}) \delta_{21} (g_{1})^{-1} \cdot v_{21} . \]

Also, it follows from the definitions of \( f' \) and \( \kappa' \) that
\[ f'(g_{1}) = k(g_{1}) f'(g) \]
\[ = \prod_{n=1}^{m} x_{n}^{e_{n}}(g_{1}) f'(g) \text{ a.e.} / \mu \text{ for each } g_{1} \in E_{m} . \]
Choose \( g_{0} \in E_{m} \) such that
\[ \left\| v_{21} \right\|^{-1} \left| f' - h_{n} \right| dv_{g_{0}} = \min_{g \in E_{m}} \left\| v_{21} \right\|^{-1} \left| f' - h_{m} \right| dv_{g} \]
and let \( a_{m} = b_{m} \cdot \kappa(g_{0}) \) and \( b_{m} \in E_{m} \). Then
\[ \int \left| f' - a_{m} k_{m} \right| d \mu = \sum_{g_{1} \in E_{m}} \int \left| f' - a_{m} k_{m} \right| dv_{g_{1}} \]
\[ = \sum_{g_{1} \in E_{m}} \left\| v_{g_{0}} \right\|^{-1} \left\| v_{g_{1}} \right\| \int \left| f' - a_{m} k_{m} \right| d \left( \delta_{g_{1} g_{0}^{-1}} \cdot v_{g_{0}} \right) \]
\[ = \sum_{g_{1} \in E_{m}} \left\| v_{g_{0}} \right\|^{-1} \left\| v_{g_{1}} \right\| \int \left| f'(g_{1} g_{0}^{-1} g) - a_{m} k_{m} (g_{1} g_{0}^{-1} g) \right| dv_{g_{0}} (g) \]
\[ = \sum_{g_{1} \in E_{m}} \left\| v_{g_{0}} \right\|^{-1} \left\| v_{g_{1}} \right\| \int \left| k_{m} (g_{1} g_{0}^{-1}) (f'(g) - a_{m} k_{m} (g)) \right| dv_{g_{0}} (g) \]
\[ = \sum_{g_{1} \in E_{m}} \left\| v_{g_{0}} \right\|^{-1} \left\| v_{g_{1}} \right\| \int \left| f' - h_{m} \right| dv_{g_{0}} \]
\[ \leq \sum_{g_{1} \in E_{m}} \int \left| f' - h_{m} \right| dv_{g_{0}} = \int \left| f' - h_{m} \right| d \mu . \]

That is, \( \int \left| f' - a_{m} k_{m} \right| d \mu \leq \int \left| f' - h_{m} \right| d \mu . \) Hence
\[ \{ a_{m} k_{m} \}_{m=1}^{\infty} = \{ a_{m} \prod_{n=1}^{m} x_{n}^{e_{n}} \}_{m=1}^{\infty} \]
converges in \( L_{1}(\mu) \) norm to \( f' \). If \( a \) is a cluster point of the sequence \( \{ a_{m} \}_{m=1}^{\infty} \), then \( \{ a_{m} \prod_{n=1}^{m} x_{n}^{e_{n}} \}_{m=1}^{\infty} \) also converges to \( f' \) in \( L_{1}(\mu) \) norm. This completes the proof.

**Lemma 2.** If \( \lim \sup_{n} r_{n} < 1 \) and \( f' = \lim_{n} \prod_{n=1}^{m} x_{n}^{e_{n}} \) as in Lemma 1, with \( |a| > 0 \), then there exists \( M \), such that \( e_{n} = 0 \) if \( n > M \). Hence \( f' = a k \) where \( k = \prod_{n=1}^{M} x_{n}^{e_{n}} \in G \).

**Proof.**
\[ \int \left| \prod_{n=1}^{m-1} x_{n}^{e_{n}} - \prod_{n=1}^{m} x_{n}^{e_{n}} \right| d \mu = \int \left| 1 - \prod_{n=1}^{m} x_{n}^{e_{n}} \right| d \mu = e_{m}(1 - r_{m}) . \]

Hence if \( \{ \prod_{n=1}^{M} x_{n}^{e_{n}} \}_{m=1}^{\infty} \) converges in \( L_{1}(\mu) \) norm, then either \( \lim \sup_{n} r_{n} = 1 \) or \( \{ e_{n} \}_{n=1}^{\infty} \) is eventually zero.
For each positive integer \( n \) let \( A_n \) be the subset of \([0, 1]\) consisting of 1 and all finite products \( \prod_{i=1}^{m} r_{n_i} \) with \( n < n_i \) for \( i = 1, 2, \ldots, m \) and \( n \neq n_j \) if \( i \neq j \).

**Lemma 3.** If \( \limsup_n r_n < 1 \), then the closure of \( \hat{G} \) in the weak-* topology of \( L_\infty(\mu) \) is \( \{ ak : k \in \hat{G} \text{ and } a \in \bigcap_n \overline{A}_n \} \).

**Proof.** If \( a \in \bigcap_n \overline{A}_n \) then there is a sequence \( \{ \{ p_{i,n} \}_{i=1}^{m_n} \}_{n=1}^\infty \) of tuples of distinct integers, with \( p_{i,n} \geq n \), such that \( \lim_n \prod_{i=1}^{m_n} r_{p_{i,n}} = a \). If \( k \in \hat{G} \) then \( k \) is a product of \( \chi_p \)'s with \( p \leq M \) for some integer \( M \); set \( h_n = k \prod_{i=1}^{m_n} \chi_{p_{i,n}} \in \hat{G} \). If \( U \) is any open-compact rectangle in \( G \) of the form \( U = \{ g \in G : \chi_{\sigma_j}(g) = \sigma_j \} \) for \( j = 1, 2, \ldots, u \) where \( \{ \sigma_j \}_{j=1}^{u} \) is any \( u \)-tuple of 1's and -1's, then \( \int_U h_n \, d\mu = \prod_{i=1}^{m_n} r_{p_{i,n}} \int_U k \, d\mu \) provided \( n > q_j \) for \( j = 1, 2, \ldots, u \), and \( n > M \). Hence \( \lim_n \int_U h_n \, d\mu = \int_U ak \, d\mu \). From this fact and the fact that \( \{ h_n \}_{n=1}^\infty \) is uniformly bounded it follows that \( \lim_n h_n = ak \) in the weak-* topology of \( L_\infty(\mu) \).

Conversely, suppose \( h \) is in the weak-* closure of \( \hat{G} \) in \( L_\infty(\mu) \). Then \( h = f' \) for some \( f \in S \) and hence, by Lemma 2, \( h = ak \) for some \( a \) with \( |a| \leq 1 \) and \( k \in \hat{G} \). Let \( \{ k_a \} \) be a net in \( \hat{G} \) converging weak-* to \( ak \). Then \( \lim_n kk_a = a \). If \( a \) is not 1 then we may assume that \( kk_a = \prod_{i=1}^{m_n} \chi_{n_i,a} \), where \( n_{i,a} \neq n_{j,a} \) if \( i \neq j \). Then \( \lim \int kk_a \, d\mu = \lim \prod_{i=1}^{m_n} r_{n_{i,a}} = a \). Also, since the weak-* limit of \( \{ kk_a \} \) is a constant, it follows that, given \( n \), eventually \( n_{i,a} \geq n \) for \( i = 1, 2, \ldots, m_a \). Hence \( a \in \bigcap_n \overline{A}_n \). This completes the proof.

**Theorem 1.** If \( \limsup_n r_n < 1 \), \( k \in \hat{G} \), and \( 0 < |a| < 1 \), then \( ak = f' \) for some \( f \) in the Shilov boundary of \( M(G) \) if and only if \( |a| \in \bigcap_n \overline{A}_n \).

**Proof.** If \( |a| \in \bigcap_n \overline{A}_n \) then, by Lemma 3, \( |a| \) is in the weak-* closure in \( L_\infty(\mu) \) of \( \hat{G} \). It follows that there exists \( h \) in the closure of \( \hat{G} \) in \( S \) such that \( h' = |a| \), that is, \( h \) is identically \( |a| \) on the carrier of \( \mu_S \) in \( S \). Then \( h \) is identically \( |a| \) on the carrier of \( \mu_S^n \) in \( S \) for each \( n \). Since \( 0 < |a| < 1 \) it follows that carrier \( \{ \mu_S^n \} \cap \text{carrier}(\mu_S^n) = \emptyset \) for \( n \neq m \). Let \( \nu(V) = \int_V |a|^{-1} ak \, d\mu \) for each Borel set \( V \) of \( G \). Then carrier \( \{ \nu_S^n \} \cap \text{carrier}(\nu_S^n) = \emptyset \) for \( n \neq m \), and hence

\[
\| (\nu + \delta_e)^n \| = \sum_{m=0}^{n} \binom{n}{m} \nu^n = \sum_{m=0}^{n} \binom{n}{m} \|\nu^m\| = \sum_{m=0}^{n} \binom{n}{m} = 2^n,
\]

where \( e \) is the identity of \( G \) and \( \delta_e \) is the point measure at \( e \). Thus \( \nu + \delta_e \) has spectral radius 2 and it follows that there exists \( h_1 \) in the Shilov boundary of \( M(G) \), such that \( |(\nu + \delta_e)(h_1)| = |\nu(h_1) + 1| = 2 \). Since \( \|\nu\| = 1 \), \( \nu(h_1) \) must be 1. Then \( \int |a|^{-1} ak h_1 \, d\mu = 1 \) and we conclude that \( h_1' = |a|^{-1} ak \) and \( (hh_1)' = ak \). Now the Shilov boundary is clearly invariant under multiplication by elements of \( \hat{G} \) and, since the
Shilov boundary is closed, it is invariant under multiplication by elements of the closure of $\hat{G}$ in $\hat{S}$. Hence $hh$ is in the Shilov boundary.

Conversely, suppose $ak = f'$ where $f$ is in the Shilov boundary. By Theorem 3.3 of [1], $f$ is the limit of a net $\{h_\alpha\} \subset H = \{h \in \hat{S}: |h(s)| = 0 \text{ or } 1 \text{ for } s \in S\}$. By Lemmas 1 and 2, there exist numbers $a_\alpha$, $|a_\alpha| = 0$ or 1, and characters $k_\alpha$, such that $h'_\alpha = a_\alpha k_\alpha$ for each $\alpha$. Clearly, $\lim a_\alpha = a/|a|$ and $\lim k_\alpha k_\alpha = |a|k$ in the weak-* topology of $L_\alpha(\mu)$. Hence, by Lemma 3, $|a| \in \cap_n \overline{A}_n$.

**Theorem 2.** The Shilov boundary of $M(G)$ is a proper subset of $\hat{S}$.

**Proof.** If $\{r_n\}_{n=1}$ is chosen such that $0 < \lim sup_n r_n < 1$, then there is a positive number $a \in \cap_n \overline{A}_n$. Then $a = f_a'$ for some $f_a \in \hat{S}$, by Theorem 1, where $f_a$ may be chosen such that $f_a(s) \geq 0$ for each $s \in S$. Hence, $f_a' \in \hat{S}$ for each complex number $z$ with $Re z > 0$, and $f_a'' = a^\ast$. It follows that for each $b$ in the unit disc there exists $f_b \in \hat{S}$, such that $f_b' = b$. By Theorem 1, $f_b$ may be chosen from the Shilov boundary if and only if $|b| \in \cap_n \overline{A}_n$. However $\cap_n \overline{A}_n \subset [0, \lim sup_n r_n] \cup 1$ which is a proper subset of $[0, 1]$. This completes the proof.

**References**


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