EXTREME HAMILTONIAN CIRCUITS. RESOLUTION OF THE CONVEX-EVEN CASE

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Let r noncollinear points in the Euclidean plane fall on the boundary B of their convex hull. It is known that the shortest polygon having these points as vertices coincides with B. In [1] the ordering of these points which yields the longest polygon is obtained for the case where r is odd. In this paper the even case is resolved.

THEOREM. *Let Σ denote a set of $2n$ noncollinear coplanar points which fall on the boundary B of their convex hull and let $P_1^1, P_1^2, \cdots, P_1^n$ denote any n points of Σ which are adjacent on B. Then, every longest polygon having precisely the points of Σ as vertices is among the n polygons*[1]

$$(1)\qquad [\cdots P^i_{2n-5}P^i_5P^i_{2n-3}P^i_3P^i_{2n-1}P^i_1P^i_{2n}P^i_2P^i_{2n-2}P^i_4P^i_{2n-4}P^i_6\cdots] \qquad (i = 1, 2, \cdots, n)$$

where for each i, starting with P_1^i and traversing B in a specified common direction the consecutive points of Σ are labeled

$$(2)\qquad P^i_1, P^i_2, P^i_3, \cdots, P^i_n, P^i_{2n}, P^i_{2n-1}, P^i_{2n-2}, \cdots, P^i_{n+1}.$$

REMARK. The collinear case (odd or even) is a special case of Theorem III [2, p. 181].

PROOF OF THE THEOREM. *Case* I. Suppose no three points of Σ are collinear. A line segment DE with endpoints in Σ is said to be of *type* L_k $(1 \leq k \leq n)$, if D and E are the endpoints of a polygonal subarc of B having k edges. For the cases where $2n$ is equal to 2 or 4, the Theorem is obviously true. Thus, in all that follows we assume $6 \leq 2n$.

Let $h = [R_1 \cdots R_{2n}]$ denote any polygon having the points of Σ as vertices and having at least one edge R_iR_{i+1} (subscripts reduced modulo $2n$) of type L_k with $1 \leq k \leq n-2$. It will be shown that, in this case, it is possible to construct a polygon longer than h. The vertices R_i and R_{i+1} define the following partition of B: $B_1 \cup B_2 \cup \{R_i, R_{i+1}\}$, where B_1 is the component of $B - \{R_i, R_{i+1}\}$ containing exactly $k-1$ points of Σ. We first show that there are at least three edges of h having both vertices in B_2 (only two of these edges will be used in

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[1] The symbols for polygons are to be considered cyclic and symmetric.

what follows). There are at most $2(k-1)+2$ edges of h incident to the vertices in $B_1 \cup \{R_i, R_{i+1}\}$ which terminate at vertices in B_2 and there are $2n-(k-1)-2$ points of Σ in B_2. Thus, there are at least the following number of edges of h which have both vertices in B_2:

$$N = \frac{2(2n-(k-1)-2)-(2(k-1)+2)}{2} = 2n-2k-1.$$

Since $k \leqq n-2$, we have $N \geqq 2n-2(n-2)-1=3$.

In order to describe how h can be transformed into a polygon of strictly greater length we introduce the following notation. The symbol $[V_1 \cdots V_{i-1}(V_i \cdots V_j)V_{j+1} \cdots V_r]$ will denote the polygon $[V_1 \cdots V_{i-1}V_jV_{j-1} \cdots V_iV_{j+1} \cdots V_r]$ and the operation $[\cdots(\cdots)\cdots]$ will be referred to as an *arcinversion* (cf. [2, p. 180]). Throughout this paper, all edges are assumed to be closed. Let PQ and RS denote disjoint directed edges with endpoints in Σ and C the boundary of the convex hull of $\{P, Q, R, S\}$. We shall say that PQ and RS have the *same* or *opposite C-sense* accordingly as they agree or conflict in inducing an orientation of C. We note that an oriented polygon having two edges with the same C-sense can be transformed into a longer polygon by an arcinversion.

Let R_jR_{j+1} and R_kR_{k+1} denote two edges of h each of which has both vertices in B_2 and let all edges of h be directed so as to agree with a fixed orientation of h. Note that the (closed) edges R_jR_{j+1} and R_kR_{k+1} may intersect. The case where they are adjacent edges is not excluded. However, they are both disjoint from R_iR_{i+1}. Thus, we may speak of the C-sense of R_iR_{i+1} with respect to R_jR_{j+1} and R_kR_{k+1} respectively. If R_iR_{i+1} has the same C-sense as either R_jR_{j+1} or R_kR_{k+1}, say R_jR_{j+1}, then the arcinversion $[\cdots R_j(R_{j+1} \cdots R_i)R_{i+1} \cdots]$ yields a polygon which is longer than h. If R_iR_{i+1} has C-sense opposite to both R_jR_{j+1} and R_kR_{k+1}, let C' denote the boundary of the convex hull of these three edges and R that terminal point of R_jR_{j+1} or R_kR_{k+1} which is adjacent to R_{i+1} on C'. Suppose $R=R_{j+1}$ (if $R=R_{k+1}$ a completely analogous situation exists). Then, consider the edge $R_{j+1}R_{j+2}$ and the partition of B into the two components B_3 and $B-B_3$, where B_3 is the closed subarc of $B-\{R_j\}$ with endpoints R_i and R_{j+1}. Note that R_{j+2} is not necessarily in B_2. Then, either (i) R_{j+2} is in B_3, $R_{j+1}R_{j+2}$ has the same C-sense as R_kR_{k+1}, and the arcinversion $[\cdots R_{j+1}(R_{j+2} \cdots R_k)R_{k+1} \cdots]$ yields a polygon which is longer than h or (ii) R_{j+2} is in $B-B_3$, $R_{j+1}R_{j+2}$ has the same C-sense as R_iR_{i+1}, and the arcinversion $[\cdots R_{j+1}(R_{j+2} \cdots R_i)R_{i+1} \cdots]$ yields a polygon which is longer than h. Therefore, a longest polygon cannot contain any edges of type L_k with $1 \leqq k \leqq n-2$.

Consider now a polygon h' which consists entirely of edges of type L_{n-1}. Note that for $2n$ vertices such a polygon exists only if $n-1$ and $2n$ are relatively prime, that is, only if n is even. The polygon h' is represented by

$$[P_1^1P_n^1P_{n+2}^1P_{n-2}^1P_{n+4}^1 \cdots P_{2n}^1P_{n+1}^1P_{n-1}^1P_{n+3}^1P_{n-3}^1 \cdots] \tag{3}$$

where the vertices are labeled as indicated in (2) with $i=1$. The arc-inversion $[P_1^1(P_n^1 \cdots P_{2n}^1)P_{n+1}^1 \cdots]$ applied to (3) yields a polygon which is longer than h'. Specifically, this arcinversion yields the polygon (1) with $i=1$, which contains two edges of type L_n and $2n-2$ edges of type L_{n-1}.

To complete the proof of Case I it remains to show: if w is any polygon containing at least one edge of type L_n and having all other edges of type L_n or of type L_{n-1}, then w is one of the n polygons indicated in the statement of the theorem. We shall refer to these n polygons as *β-polygons.*

We select any edge of type L_n in w, label it W_1W_{2n}, and call it the first edge of w. For the second edge we select the other edge incident to W_{2n}, label it $W_{2n}W_2$ and note that it is of type L_{n-1}. The rest of the vertices of w can now be labeled

$$W_1, W_2, W_3, \cdots, W_n, W_{2n}, W_{2n-1}, W_{2n-2}, \cdots, W_{n+1}$$

in cyclic order around the boundary B of their convex hull in a unique way compatible with the three vertices labeled thus far. Orienting w by W_1W_{2n}, we note that all odd numbered edges will terminate in vertices having subscript greater than n, and all even numbered edges will terminate in vertices having subscript less than or equal to n. Thus, edge $2n$ must be $W_{2n-1}W_1$. Since $6 \leqq 2n$, the third edge of w must be the edge W_2W_{2n-2} and edge $2n-1$ must be the edge W_3W_{2n-1}. Both of these edges are of type L_{n-1}.

The fourth edge must either be of type L_n and completes w (if and only if $n=3$), or must be the edge $W_{2n-2}W_4$. If $n>3$, then edge $2n-2$ must be the edge $W_{2n-3}W_3$.

We proceed in this manner, repeating the above argument until the vertices W_n and W_{n+1} are joined by an edge of type L_n. This edge completes w which is seen to be of the form (1) with W in place of P^i.

If W_1W_2 and $P_1^1P_2^1$ induce the same orientation of B and $W_1=P_1^i$ for some $i\in\{1, 2, \cdots, n\}$, then w is a β-polygon.

Suppose that W_1W_2 and $P_1^1P_2^1$ induce the same orientation of B but $W_1\neq P_1^i$ for some $i\in\{1, 2, \cdots, n\}$. Then,

$$W_i' = W_{2n+1-i} \qquad (i = 1, 2, \cdots, 2n)$$

defines a relabeling of the vertices of w such that w is of the form (1) with W' in place of P^i, $W_1' W_2'$ and $P_1^1P_2^1$ induce the same orientation of B, and $W_1' = P_1^i$ for some $i \in \{1, 2, \cdots, n\}$. Thus, w is a β-polygon.

Suppose that W_1W_2 and $P_1^1P_2^1$ induce opposite orientations of B. In this case,

$$W_i' = \begin{cases} W_{n+i} & (i = 1, 2, \cdots, n), \\ W_{i-n} & (i = n+1, n+2, \cdots, 2n) \end{cases}$$

defines a relabeling of the vertices of w such that w is of the form (1) with W' in place of P^i, and $W_1' W_2'$ and $P_1^1P_2^1$ induce the same orientation of B. Thus, by the preceding two paragraphs, w is a β-polygon.

Case II. Suppose B has support lines passing through at least three points of Σ. Let X be a point in the interior of the convex hull of Σ and $B(t)$ $(0 \leqq t < 1)$ be a family of strongly convex curves circumscribing B and converging to B as t approaches 1. Let $P_j^i(t)$ be the intersection of $B(t)$ with the ray emanating from X and passing through P_j^i $(1 \leqq j \leqq 2n; 1 \leqq i \leqq n)$ and let $\Sigma(t) \equiv \{P_j^i(t)\}$. Then, for each t $(0 \leqq t < 1)$ Case I implies that every longest polygon having the points of $\Sigma(t)$ as vertices is a $\beta(t)$-polygon. Now, for t sufficiently close to 1, a polygon having $P_j^i(t)$ $(1 \leqq j \leqq 2n$; i fixed) as vertices is arbitrarily close to the corresponding polygon having the P_j^i's as vertices. Therefore, every longest polygon with vertices in B is a β-polygon.

REMARK. If the points of Σ are evenly distributed on a circle, then the n β-polygons have the same length. On the other hand, sets Σ (whose points all fall on the boundary of their convex hull) may be selected such that no two polygons having the points of Σ as their vertices are equal in length.

REFERENCES

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