EXTREME HAMILTONIAN CIRCUITS. RESOLUTION OF THE CONVEX-EVEN CASE

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Let \( r \) noncollinear points in the Euclidean plane fall on the boundary \( B \) of their convex hull. It is known that the shortest polygon having these points as vertices coincides with \( B \). In [1] the ordering of these points which yields the longest polygon is obtained for the case where \( r \) is odd. In this paper the even case is resolved.

Theorem. Let \( \Sigma \) denote a set of \( 2n \) noncollinear coplanar points which fall on the boundary \( B \) of their convex hull and let \( P_1, P_2, \ldots, P_n \) denote any \( n \) points of \( \Sigma \) which are adjacent on \( B \). Then, every longest polygon having precisely the points of \( \Sigma \) as vertices is among the \( n \) polygons

\[
\{ \ldots P_{2n-4}^i P_{2n-3}^i P_{2n-2}^i P_{2n-1}^i P_{2n}^i P_{2n-2}^i P_{2n-3}^i P_{2n-4}^i P_{2n-5}^i P_{2n-6}^i \ldots \}
\]

(\( i = 1, 2, \ldots, n \))

where for each \( i \), starting with \( P_1^i \) and traversing \( B \) in a specified common direction the consecutive points of \( \Sigma \) are labeled

\[
P_1^i, P_2^i, P_3^i, \ldots, P_n^i, P_{2n}^i, P_{2n-1}^i, P_{2n-2}^i, \ldots, P_{n+1}^i.
\]

Remark. The collinear case (odd or even) is a special case of Theorem III [2, p. 181].

Proof of the Theorem. Case I. Suppose no three points of \( \Sigma \) are collinear. A line segment \( DE \) with endpoints in \( \Sigma \) is said to be of type \( L_k \) (1 \( \leq k \leq n \)), if \( D \) and \( E \) are the endpoints of a polygonal sub-arc of \( B \) having \( k \) edges. For the cases where \( 2n \) is equal to 2 or 4, the Theorem is obviously true. Thus, in all that follows we assume \( 6 \leq 2n \).

Let \( h = [R_1 \cdots R_{2n}] \) denote any polygon having the points of \( \Sigma \) as vertices and having at least one edge \( R_i R_{i+1} \) (subscripts reduced modulo \( 2n \)) of type \( L_k \) with \( 1 \leq k \leq n - 2 \). It will be shown that, in this case, it is possible to construct a polygon longer than \( h \). The vertices \( R_i \) and \( R_{i+1} \) define the following partition of \( B \): \( B_1 \cup B_2 \cup \{ R_i, R_{i+1} \} \), where \( B_1 \) is the component of \( B - \{ R_i, R_{i+1} \} \) containing exactly \( k - 1 \) points of \( \Sigma \). We first show that there are at least three edges of \( h \) having both vertices in \( B_2 \) (only two of these edges will be used in

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\(^1\) The symbols for polygons are to be considered cyclic and symmetric.

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what follows). There are at most \(2(k - 1) + 2\) edges of \(h\) incident to the vertices in \(B_i \cup \{R_i, R_{i+1}\}\) which terminate at vertices in \(B_2\) and there are \(2n - (k - 1) - 2\) points of \(\Sigma\) in \(B_3\). Thus, there are at least the following number of edges of \(h\) which have both vertices in \(B_2\):

\[
N = \frac{2(2n - (k - 1) - 2) - (2(k - 1) + 2)}{2} = 2n - 2k - 1.
\]

Since \(k \leq n - 2\), we have \(N \geq 2n - 2(n - 2) - 1 = 3\).

In order to describe how \(h\) can be transformed into a polygon of strictly greater length we introduce the following notation. The symbol \([V_1 \cdots V_{i-1}(V_i \cdots V_j)V_{j+1} \cdots V_r]\) will denote the polygon \([V_1 \cdots V_{i-1}V_jV_{j-1} \cdots V_iV_{j+1} \cdots V_r]\) and the operation \([\cdots (\cdots ) \cdots]\) will be referred to as an arcinversion (cf. [2, p. 180]). Throughout this paper, all edges are assumed to be closed. Let \(PQ\) and \(RS\) denote disjoint directed edges with endpoints in \(\Sigma\) and \(C\) the boundary of the convex hull of \(\{P, Q, R, S\}\). We shall say that \(PQ\) and \(RS\) have the same or opposite \(C\)-sense accordingly as they agree or conflict in inducing an orientation of \(C\). We note that an oriented polygon having two edges with the same \(C\)-sense can be transformed into a longer polygon by an arcinversion.

Let \(R_iR_{i+1}\) and \(R_kR_{k+1}\) denote two edges of \(h\) each of which has both vertices in \(B_2\) and let all edges of \(h\) be directed so as to agree with a fixed orientation of \(h\). Note that the (closed) edges \(R_iR_{i+1}\) and \(R_kR_{k+1}\) may intersect. The case where they are adjacent edges is not excluded. However, they are both disjoint from \(R_iR_{i+1}\) and \(R_kR_{k+1}\) respectively. If \(R_iR_{i+1}\) has the same \(C\)-sense as either \(R_jR_{j+1}\) or \(R_kR_{k+1}\), say \(R_jR_{j+1}\), then the arcversion \([\cdots R_j(R_{j+1} \cdots R_i)R_{i+1} \cdots]\) yields a polygon which is longer than \(h\). If \(R_iR_{i+1}\) has \(C\)-sense opposite to both \(R_jR_{j+1}\) and \(R_kR_{k+1}\), let \(C\) denote the boundary of the convex hull of these three edges and \(R\) that terminal point of \(R_jR_{j+1}\) or \(R_kR_{k+1}\) which is adjacent to \(R_{i+1}\) on \(C\). Suppose \(R = R_{j+1}\) (if \(R = R_{k+1}\) a completely analogous situation exists). Then, consider the edge \(R_{j+1}R_{j+2}\) and the partition of \(B\) into the two components \(B_3\) and \(B - B_3\), where \(B_3\) is the closed subarc of \(B - \{R_i\}\) with endpoints \(R_i\) and \(R_{j+1}\). Note that \(R_{j+2}\) is not necessarily in \(B_3\). Then, either (i) \(R_{j+2}\) is in \(B_3\), \(R_{j+1}R_{j+2}\) has the same \(C\)-sense as \(R_kR_{k+1}\), and the arcversion \([\cdots R_{j+1}(R_{j+2} \cdots R_i)R_{k+1} \cdots]\) yields a polygon which is longer than \(h\) or (ii) \(R_{j+2}\) is in \(B - B_3\), \(R_{j+1}R_{j+2}\) has the same \(C\)-sense as \(R_kR_{k+1}\), and the arcversion \([\cdots R_{j+1}(R_{j+2} \cdots R_i)R_{i+1} \cdots]\) yields a polygon which is longer than \(h\). Therefore, a longest polygon cannot contain any edges of type \(L_k\) with \(1 \leq k \leq n - 2\).
Consider now a polygon $h'$ which consists entirely of edges of type $L_{n-1}$. Note that for $2n$ vertices such a polygon exists only if $n - 1$ and $2n$ are relatively prime, that is, only if $n$ is even. The polygon $h'$ is represented by

$$\left[ P_1^1 P_n^1 P_{n+2}^1 P_{n+4}^1 \cdots P_{2n}^1 P_{n+1}^1 P_{n+3}^1 P_{n+5}^1 \cdots \right]$$

where the vertices are labeled as indicated in (2) with $i = 1$. The arc-inversion $[P_i^1 (P_n^1 \cdots P_{2n}^1) P_{i+1}^1 \cdots ]$ applied to (3) yields a polygon which is longer than $h'$. Specifically, this arc-inversion yields the polygon (1) with $i = 1$, which contains two edges of type $L_n$ and $2n - 2$ edges of type $L_{n-1}$.

To complete the proof of Case I it remains to show: if $w$ is any polygon containing at least one edge of type $L_n$ and having all other edges of type $L_n$ or of type $L_{n-1}$, then $w$ is one of the $n$ polygons indicated in the statement of the theorem. We shall refer to these $n$ polygons as $\beta$-polygons.

We select any edge of type $L_n$ in $w$, label it $W_1 W_{2n}$, and call it the first edge of $w$. For the second edge we select the other edge incident to $W_{2n}$, label it $W_{2n} W_1$ and note that it is of type $L_{n-1}$. The rest of the vertices of $w$ can now be labeled

$$W_1, W_2, W_3, \ldots, W_n, W_{2n}, W_{2n-1}, W_{2n-2}, \ldots, W_{n+1}$$

in cyclic order around the boundary $B$ of their convex hull in a unique way compatible with the three vertices labeled thus far. Orienting $w$ by $W_1 W_{2n}$, we note that all odd numbered edges will terminate in vertices having subscript greater than $n$, and all even numbered edges will terminate in vertices having subscript less than or equal to $n$. Thus, edge $2n$ must be $W_{2n-1} W_1$. Since $6 \leq 2n$, the third edge of $w$ must be the edge $W_2 W_{2n-2}$ and edge $2n - 1$ must be the edge $W_3 W_{2n-3}$. Both of these edges are of type $L_{n-1}$.

The fourth edge must either be of type $L_n$ and completes $w$ (if and only if $n = 3$), or must be the edge $W_{2n-2} W_4$. If $n > 3$, then edge $2n - 2$ must be the edge $W_{2n-3} W_3$.

We proceed in this manner, repeating the above argument until the vertices $W_n$ and $W_{n+1}$ are joined by an edge of type $L_n$. This edge completes $w$ which is seen to be of the form (1) with $P_i$ in place of $P^i$. If $W_1 W_2$ and $P_1^i P_2^i$ induce the same orientation of $B$ and $W_1 = P_1^i$ for some $i \in \{1, 2, \cdots, n\}$, then $w$ is a $\beta$-polygon.

Suppose that $W_1 W_2$ and $P_1^i P_2^i$ induce the same orientation of $B$ but $W_1 \neq P_1^i$ for some $i \in \{1, 2, \cdots, n\}$. Then,

$$W_i^i = W_{2n+1-i} \quad (i = 1, 2, \cdots, 2n)$$
defines a relabeling of the vertices of \( w \) such that \( w \) is of the form (1) with \( W' \) in place of \( P_i \), \( W'_1 W'_2 \) and \( P'_1 P'_2 \) induce the same orientation of \( B \), and \( W'_1 = P'_1 \) for some \( i \in \{ 1, 2, \ldots, n \} \). Thus, \( w \) is a \( \beta \)-polygon.

Suppose that \( W_1 W_2 \) and \( P_1 P_2 \) induce opposite orientations of \( B \). In this case,

\[
W'_i = \begin{cases} 
W_{n+i} & (i = 1, 2, \ldots, n), \\
W_{i-n} & (i = n + 1, n + 2, \ldots, 2n)
\end{cases}
\]

defines a relabeling of the vertices of \( w \) such that \( w \) is of the form (1) with \( W' \) in place of \( P_i \), and \( W'_1 W'_2 \) and \( P'_1 P'_2 \) induce the same orientation of \( B \). Thus, by the preceding two paragraphs, \( w \) is a \( \beta \)-polygon.

**Case II.** Suppose \( B \) has support lines passing through at least three points of \( \Sigma \). Let \( X \) be a point in the interior of the convex hull of \( \Sigma \) and \( B(t) \) \((0 \leq t < 1)\) be a family of strongly convex curves circumscribing \( B \) and converging to \( B \) as \( t \) approaches 1. Let \( P_j(t) \) be the intersection of \( B(t) \) with the ray emanating from \( X \) and passing through \( P_j \) \((1 \leq j \leq 2n; 1 \leq i \leq n)\) and let \( \Sigma(t) = \{ P_j(t) \} \). Then, for each \( t \) \((0 \leq t < 1)\) Case I implies that every longest polygon having the points of \( \Sigma(t) \) as vertices is a \( \beta(t) \)-polygon. Now, for \( t \) sufficiently close to 1, a polygon having \( P_j(t) \) \((1 \leq j \leq 2n; i \) fixed) as vertices is arbitrarily close to the corresponding polygon having the \( P_j \)'s as vertices. Therefore, every longest polygon with vertices in \( B \) is a \( \beta \)-polygon.

**Remark.** If the points of \( \Sigma \) are evenly distributed on a circle, then the \( n \) \( \beta \)-polygons have the same length. On the other hand, sets \( \Sigma \) (whose points all fall on the boundary of their convex hull) may be selected such that no two polygons having the points of \( \Sigma \) as their vertices are equal in length.

**References**


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