THE SCHNIRELMANN DENSITY OF THE k-FREE INTEGERS

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Let T_k denote the set of k-free integers and let $T_k(n)$ be the number of such numbers not exceeding n. The Schnirelmann density of T_k is defined by

$$d(T_k) = \inf \frac{T_k(n)}{n} \cdot \tag{1}$$

Since $T_2 \subset T_3 \subset \cdots \subset T_k \subset \cdots$, it is clear that

$$d(T_2) \leqq d(T_3) \leqq \cdots \leqq d(T_k) \leqq \cdots \leqq 1. \tag{2}$$

It has been shown recently by Rogers [1] that $d(T_2) = 53/88$ and it was also observed that this value is less than that of the asymptotic density which is given for T_k by

$$\delta(T_k) = \lim_{n\to\infty} \frac{T_k(n)}{n} = \frac{1}{\zeta(k)}, \tag{3}$$

where $\zeta(s)$ is the Riemann zeta function. An elementary proof of a much stronger asymptotic formula for $T_k(n)$ has been given recently by Cohen [2].

Such an explicit determination of $d(T_k)$ as a function of k appears to be a difficult problem. We content ourselves here with the following refinement of (2). It is clear that

$$T_k(n) \geqq n - \sum_p \left[\frac{n}{p^k}\right] \qquad \text{and} \qquad \frac{T_k(n)}{n} > 1 - \sum_p p^{-k}.$$

Thus we have the following

LEMMA. $d(T_k) > 1 - \sum_p p^{-k}$.

THEOREM. $\delta(T_k) < d(T_{k+1}) \leqq \delta(T_{k+1})$.

PROOF. The second inequality follows from (1) and (3). By the Lemma, the first inequality will be true if

$$\frac{1}{\zeta(k)} + \sum_p p^{-k-1} \leqq 1.$$

Received by the editors September 14, 1964.

Since

$$\sum_p p^{-k-1} < \zeta(k+1) - 1,$$

it will be sufficient to show that

$$\frac{1}{\zeta(k)} + \zeta(k+1) \leqq 2.$$

Let

$$g(s) = \frac{1}{\zeta(s)} + \zeta(s+1).$$

Then, since

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=2}^{\infty} \Lambda(n) n^{-s},$$

$$g'(s) = \zeta'(s+1) - \frac{\zeta'(s)}{\zeta^2(s)} = \sum_{n=2}^{\infty} \left\{ \frac{n}{\zeta(s)} - \zeta(s+1) \right\} \Lambda(n) n^{-s-1}.$$

Thus $g'(s) > 0$ if $\zeta(s)\zeta(s+1) < 2$. But $\zeta(s)\zeta(s+1)$ is a decreasing function and $\zeta(2)\zeta(3) < 2$. Hence $g'(s) > 0$ and $g(s)$ is an increasing function for $s \geqq 2$. The desired result now follows since

$$\lim_{s \to \infty} g(s) = 2.$$

COROLLARY 1. $d(T_k) < d(T_{k+1})$.

COROLLARY 2. $\lim_{k\to\infty} d(T_k) = 1$.

REFERENCES

1. Kenneth Rogers, *The Schnirelmann density of the squarefree integers*, Proc. Amer. Math. Soc. **15** (1964), 515–516.

2. Eckford Cohen, *An elementary estimate for the k-free integers*, Bull. Amer. Math. Soc. **69** (1963), 762–765.

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