## REFERENCE

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# ON PIECEWISE LINEAR IMMERSIONS

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The purpose of this note is to prove an existence theorem for immersions of piecewise linear manifolds in Euclidean space. A more comprehensive theory of piecewise linear immersions has been worked out by Haefliger and Poenaru [**1**].

All maps, manifolds, microbundles, etc. are piecewise linear unless the contrary is explicitly indicated.

Let $M$ be a manifold without boundary, of dimension $n$. Denote the tangent microbundle of $M$ by $\tau_M$, and the trivial microbundle over $M$ of (fibre) dimension $k$ by $\epsilon^k$. Let

$$\nu: M \xrightarrow{i} E \xrightarrow{j} M$$

be a microbundle of dimension $k$ such that $E$ is a manifold. An *immersion* of $M$ in $R^{n+k}$ is a locally one-one map $f: M \to R^{n+k}$.

I say $f$ has a *normal bundle of type* $\nu$ if there is an immersion $g: E \to R^{n+k}$ such that $gi = f$. (It is unknown whether $f$ necessarily has a normal bundle, or whether all normal bundles of $f$ are of the same type.)

The converse of the following theorem is trivial.

THEOREM. *Assume that if $k=0$, then $M$ has no compact component. There exists an immersion of $M$ in $R^{n+k}$ having a normal bundle of type $\nu$ if there exists an isomorphism*

$$\phi: \tau_m \oplus \nu \to \epsilon^{n+k}$$

PROOF. We may assume that $i(M)$ is a deformation retract of the total space $E$ of $\nu$. By Milnor [**3**], $\tau_E \mid i(M)$ is isomorphic to $\tau_M \oplus \nu$; it follows from the existence of $\phi$ that $\tau_E$ is trivial. According to [**3**]

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there is a parallelizable differential structure $\alpha$ on $E$ compatible with the piecewise linear structure. Let $h: E_\alpha \to R^{n+k}$ be a differentiable immersion, which exists by Hirsch [2] or Poenaru [4]. (If $k=0$, the assumption that $M$ has no compact component is used here.) Approximate $h$ by a piecewise linear immersion $g: E \to R^{n+k}$, using the theory of $C^1$ complexes of Whitehead [5]. Clearly $gi: M \to R^{n+k}$ is an immersion having a normal bundle of type $\nu$.

REMARKS. (1) The assumption that $M$ is unbounded is unnecessary, since a bounded manifold can be embedded in its interior. However, $\tau_M$ must be redefined if $M$ has a boundary.

(2) It is not hard to define the concepts of "immersion plus normal bundle"—essentially an immersion of $E$—and of a "regular homotopy" of these; one can then prove a uniqueness theorem.

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