ON THE CONTINUOUS FUNCTION SPACE OF A BASICALLY DISCONNECTED SPACE[1]

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Throughout this note we shall let H be a Hausdorff space and let $C(H)$ be the space of bounded continuous real-valued functions on H, $C(H)$ having the usual supremum norm. Certain results (cf. e.g. [7]) suggest the possibility of showing that if a normed linear space X is complemented in every superspace (cf. [1, pp. 94 and 120]), then X is isomorphic to some space $C(H)$ over a Stone space H, and that if X is isometric to some $C(H)$, then H is basically disconnected. The purpose of this note is to extend a result of Dean [2, p. 391] for $C(H)$ where H is extremally disconnected and compact to the case where H is basically disconnected and normal. Our proof rests on an extension of James' technique [6, p. 900] for embedding the space (m) of bounded sequences in $C(H)$ where H is infinite, extremally disconnected and compact.

Let r be a real number. We shall say that H is basically disconnected if and only if the closure of every open set of the form

$$G(f, r) = \{h: f(h) < r, h \in H, f \in C(H)\}$$

is open. We note that an extremally disconnected space is basically disconnected, that a basically disconnected space is totally disconnected, and that in a normal space an open set is an F_σ set if and only if it is a set of the form $G(f, r)$ (cf. [3, p. 15]).

Our first lemma contains a result of Dean [2, p. 391].

LEMMA 1. *If H is an infinite basically disconnected normal Hausdorff space, if W is an infinite open and closed subset of H, and if h' is a point in W, then $W-\{h'\}$ contains an infinite open and closed set.*

PROOF. Suppose $W-N$ is finite whenever $N \subset W$ is a neighborhood of h'. Then each point $h \neq h'$ in W is open. Hence each countably infinite subset H' of $W-\{h'\}$ is an open F_σ set and its closure $\overline{H}'=H'\cup\{h'\}$ is open. It follows that if H' and H'' are countably infinite subsets of $W-\{h'\}$, then $\overline{H}'\cap\overline{H}''\supset\{h'\}\neq\emptyset$ even though H' and H'' may be disjoint. But this is impossible because in a basically disconnected normal Hausdorff space, disjoint open F_σ sets

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have disjoint closures. Hence there is a neighborhood N' of h' such that $N' \subset W$ and $W-N'$ is infinite. If $f \in C(H)$ takes the value 0 on $W-N'$, 1 at h', and 1 on $H-W$, the closure of $\{h: f(h) < \frac{1}{2}\}$ is an infinite open and closed subset of $W-\{h'\}$.

LEMMA 2. *If H is an infinite basically disconnected normal Hausdorff space, then H contains an infinite sequence $\{V_i\}$ of pairwise disjoint, nonempty, open and closed sets. If V is the closure of $\bigcup_{i=1}^{\infty} V_i$, then V is open and closed.*

PROOF. We will construct the sequence inductively. Let h_1 be a point in H. By Lemma 1 there is an infinite open and closed subset $N_1 \subset H-\{h_1\}$. Let $V_1 = H - N_1$. We note that $h_1 \in V_1$ and V_1 is open and closed. Suppose we have chosen pairwise disjoint open and closed sets $V_1, V_2, \cdots, V_k$ so that $h_i \in V_i$ and $N_k = H - \bigcup_{i=1}^{k} V_i$ is an infinite open and closed subset of H. Let h_{k+1} be any point in N_k. Then by Lemma 1 there is an infinite open and closed set $N_{k+1} \subset N_k - \{h_{k+1}\}$. Let $V_{k+1} = N_k - N_{k+1}$. Then $h_{k+1} \in V_{k+1}$ and V_{k+1} is open and closed. This completes the inductive construction.

Since each V_i is an open F_σ set, $\bigcup_{i=1}^{\infty} V_i$ is an open F_σ set and it follows that V is open and closed. This completes the proof.

James [6, p. 900] embedded the space (m) of bounded sequences in $C(H)$, H an infinite extremally disconnected compact Hausdorff space, by using an infinite sequence of pairwise disjoint open and closed subsets of H. Using Lemma 2 and James' procedure, we may embed (m) in $C(H)$ where H is an infinite basically disconnected normal Hausdorff space (cf. [5, p. 257]). If $\{h_i\}_{i=1}^{\infty}$, $\{V_i\}_{i=1}^{\infty}$ and V are as in Lemma 2, a suitable embedding, Q, may be defined by $Q(x) = f$ implies

$$f(h) = \begin{cases} 0 & \text{if } h \in H - V, \\ x(i) & \text{if } h \in V_i, \end{cases}$$

where $x \in (m)$ and $f \in C(H)$.

Our theorem contains a result of Dean [2, p. 391].

THEOREM. *Let H be an infinite basically disconnected normal Hausdorff space and let the space (m) of bounded sequences be embedded in $C(H)$ as above; that is, let $Q((m)) = (m') \subset C(H)$. Then a subspace B of $C(H)$ complementary to (m') is isomorphic to $C(H)$ or is finite dimensional.*

PROOF. Let $f \in C(H)$. Define Tf to be the element of (m') for which $Tf(h) = f(h_i)$ for h and h_i in V_i as in Lemma 2. Then T is a projection of $C(H)$ onto (m'), and $C(H)$ is the direct sum of (m') and the null space Y of T; that is, $C(H)=Y\oplus(m')$ (cf. [4, p. 91], [8, p. 538]). If the set H' of points in H and not in the closure of $\bigcup_{i=1}^{\infty}\{h_i\}$ is finite, then Y is finite dimensional (cf. [2, p. 392]).

If H' is infinite, then H' contains an infinite open and closed subset H''. For suppose each V_i is finite. Then each h_i is open and $\bigcup_{i=1}^{\infty}\{h_i\}$ is an open F_σ set. It follows that the closure of $\bigcup_{i=1}^{\infty}\{h_i\}$ is open and, hence, that H' itself is an infinite open and closed set. Alternatively, suppose some V_i is infinite. Then, by Lemma 1, $V_i-\{h_i\}$ contains an infinite open and closed subset H''. In either case, by Lemma 2, H'' contains an infinite sequence $\{V_i'\}_{i=1}^{\infty}$ of nonempty, pairwise disjoint, open and closed subsets. Let (m'') be the embedding of (m) in $C(H)$ determined by the sequence $\{V_i'\}_{i=1}^{\infty}$. Then (m'') is a subspace of $Y=Z\oplus(m'')$ and $C(H)=Z\oplus(m')\oplus(m'')$.

Let J be an isomorphism of (m'') onto $(m')\oplus(m'')$. Define M on Y to $C(H)$ by $M(z+x'')=z+Jx''1$ for every z in Z and x'' in (m''). Then M is an isomorphism of Y with $C(H)$ (cf. [2, p. 391]).

We have shown that Y is either finite dimensional or isomorphic to $C(H)$. To complete the proof, we need only show that B and Y are isomorphic.

Since both B and Y complement (m') in $C(H)$, $C(H)=B\oplus(m')$ $=Y\oplus(m')$. Let $P=I-T$ where I is the identity transformation of $C(H)$ onto $C(H)$ and T is the projection defined above. Since $Px'=0$ for x' in (m'), $PB=Y$. If $Pb=0$ for b in B, then b is also in (m'), and it follows that $b=0$. Hence P is an isomorphism of B with Y, which completes the proof.

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