ON THE CONTINUOUS FUNCTION SPACE OF A 
BASICALLY DISCONNECTED SPACE

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Throughout this note we shall let $H$ be a Hausdorff space and let $C(H)$ be the space of bounded continuous real-valued functions on $H$, $C(H)$ having the usual supremum norm. Certain results (cf. e.g. [7]) suggest the possibility of showing that if a normed linear space $X$ is complemented in every superspace (cf. [1, pp. 94 and 120]), then $X$ is isomorphic to some space $C(H)$ over a Stone space $H$, and that if $X$ is isometric to some $C(H)$, then $H$ is basically disconnected. The purpose of this note is to extend a result of Dean [2, p. 391] for $C(H)$ where $H$ is extremally disconnected and compact to the case where $H$ is basically disconnected and normal. Our proof rests on an extension of James' technique [6, p. 900] for embedding the space $(m)$ of bounded sequences in $C(H)$ where $H$ is infinite, extremally disconnected and compact.

Let $r$ be a real number. We shall say that $H$ is basically disconnected if and only if the closure of every open set of the form

$$G(f, r) = \{h : f(h) < r, h \in H, f \in C(H)\}$$

is open. We note that an extremally disconnected space is basically disconnected, that a basically disconnected space is totally disconnected, and that in a normal space an open set is an $F_\sigma$ set if and only if it is a set of the form $G(f, r)$ (cf. [3, p. 15]).

Our first lemma contains a result of Dean [2, p. 391].

**Lemma 1.** If $H$ is an infinite basically disconnected normal Hausdorff space, if $W$ is an infinite open and closed subset of $H$, and if $h'$ is a point in $W$, then $W - \{h'\}$ contains an infinite open and closed set.

**Proof.** Suppose $W - N$ is finite whenever $N \subseteq W$ is a neighborhood of $h'$. Then each point $h \neq h'$ in $W$ is open. Hence each countably infinite subset $H'$ of $W - \{h'\}$ is an open $F_\sigma$ set and its closure $\overline{H'} = H' \cup \{h'\}$ is open. It follows that if $H'$ and $H''$ are countably infinite subsets of $W - \{h'\}$, then $\overline{H'} \cap \overline{H''} \supset \{h'\} \neq \emptyset$ even though $H'$ and $H''$ may be disjoint. But this is impossible because in a basically disconnected normal Hausdorff space, disjoint open $F_\sigma$ sets...
have disjoint closures. Hence there is a neighborhood \( N' \) of \( h' \) such that \( N' \subseteq W \) and \( W - N' \) is infinite. If \( f \in C(H) \) takes the value 0 on \( W - N' \), 1 at \( h' \), and 1 on \( H - W \), the closure of \( \{ h : f(h) < \frac{1}{2} \} \) is an infinite open and closed subset of \( W - \{ h' \} \).

**Lemma 2.** If \( H \) is an infinite basically disconnected normal Hausdorff space, then \( H \) contains an infinite sequence \( \{ V_i \} \) of pairwise disjoint, nonempty, open and closed sets. If \( V \) is the closure of \( \bigcup_{i=1}^{\infty} V_i \), then \( V \) is open and closed.

**Proof.** We will construct the sequence inductively. Let \( h_1 \) be a point in \( H \). By Lemma 1 there is an infinite open and closed subset \( N_1 \subseteq H - \{ h_1 \} \). Let \( V_1 = H - N_1 \). We note that \( h_1 \in V_1 \) and \( V_1 \) is open and closed. Suppose we have chosen pairwise disjoint open and closed sets \( V_1, V_2, \ldots, V_k \) so that \( h_i \in V_i \), and \( N_k = H - \bigcup_{i=1}^{k} V_i \) is an infinite open and closed subset of \( H \). Let \( h_{k+1} \) be any point in \( N_k \). Then by Lemma 1 there is an infinite open and closed set \( N_{k+1} \subseteq N_k - \{ h_{k+1} \} \). Let \( V_{k+1} = N_k - N_{k+1} \). Then \( h_{k+1} \in V_{k+1} \) and \( V_{k+1} \) is open and closed. This completes the inductive construction.

Since each \( V_i \) is an open \( F_\sigma \) set, \( \bigcup_{i=1}^{\infty} V_i \) is an open \( F_\sigma \) set and it follows that \( V \) is open and closed. This completes the proof.

James [6, p. 900] embedded the space \((m)\) of bounded sequences in \( C(H) \), \( H \) an infinite extremally disconnected compact Hausdorff space, by using an infinite sequence of pairwise disjoint open and closed subsets of \( H \). Using Lemma 2 and James' procedure, we may embed \((m)\) in \( C(H) \) where \( H \) is an infinite basically disconnected normal Hausdorff space (cf. [5, p. 257]). If \( \{ h_i \}_{i=1}^{\infty}, \{ V_i \}_{i=1}^{\infty} \) and \( V \) are as in Lemma 2, a suitable embedding, \( Q \), may be defined by \( Q(x) = f \) implies

\[
Q(x)(h) = \begin{cases} 
0 & \text{if } h \in H - V, \\
x(i) & \text{if } h \in V_i,
\end{cases}
\]

where \( x \in (m) \) and \( f \in C(H) \).

Our theorem contains a result of Dean [2, p. 391].

**Theorem.** Let \( H \) be an infinite basically disconnected normal Hausdorff space and let the space \((m)\) of bounded sequences be embedded in \( C(H) \) as above; that is, let \( Q((m)) = (m') \subseteq C(H) \). Then a subspace \( B \) of \( C(H) \) complementary to \((m')\) is isomorphic to \( C(H) \) or is finite dimensional.

**Proof.** Let \( f \in C(H) \). Define \( Tf \) to be the element of \((m')\) for which \( Tf(h_i) = f(h_i) \) for \( h_i \) and \( h_i \) in \( V_i \) as in Lemma 2. Then \( T \) is a projection of \( C(H) \) onto \((m')\), and \( C(H) \) is the direct sum of \((m')\) and the null.
space $Y$ of $T$; that is, $C(H) = Y \oplus (m')$ (cf. [4, p. 91], [8, p. 538]). If the set $H'$ of points in $H$ and not in the closure of $\cup_{i=1}^{n} \{h_i\}$ is finite, then $Y$ is finite dimensional (cf. [2, p. 392]).

If $H'$ is infinite, then $H'$ contains an infinite open and closed subset $H''$. For suppose each $V_i$ is finite. Then each $h_i$ is open and $\cup_{i=1}^{n} \{h_i\}$ is an open $F_{\sigma}$ set. It follows that the closure of $\cup_{i=1}^{n} \{h_i\}$ is open and, hence, that $H'$ itself is an infinite open and closed set. Alternatively, suppose some $V_i$ is infinite. Then, by Lemma 1, $V_i - \{h_i\}$ contains an infinite open and closed subset $H''$. In either case, by Lemma 2, $H''$ contains an infinite sequence $\{V'_i\}_{i=1}^{\infty}$ of nonempty, pairwise disjoint, open and closed subsets. Let $(m'')$ be the embedding of $(m)$ in $C(H)$ determined by the sequence $\{V'_i\}_{i=1}^{\infty}$. Then $(m'')$ is a subspace of $Y = Z \oplus (m'')$ and $C(H) = Z \oplus (m') \oplus (m'')$.

Let $J$ be an isomorphism of $(m'')$ onto $(m') \oplus (m'')$. Define $M$ on $Y$ to $C(H)$ by $M(z + x'') = z + Jx''$ for every $z$ in $Z$ and $x''$ in $(m'')$. Then $M$ is an isomorphism of $Y$ with $C(H)$ (cf. [2, p. 391]).

We have shown that $Y$ is either finite dimensional or isomorphic to $C(H)$. To complete the proof, we need only show that $B$ and $Y$ are isomorphic.

Since both $B$ and $Y$ complement $(m')$ in $C(H)$, $C(H) = B \oplus (m') = Y \oplus (m')$. Let $P = I - T$ where $I$ is the identity transformation of $C(H)$ onto $C(H)$ and $T$ is the projection defined above. Since $Px' = 0$ for $x'$ in $(m')$, $PB = Y$. If $Pb = 0$ for $b$ in $B$, then $b$ is also in $(m')$, and it follows that $b = 0$. Hence $P$ is an isomorphism of $B$ with $Y$, which completes the proof.

References

6. R. C. James, Projections in the space $(m)$, Proc. Amer. Math. Soc. 6 (1955), 899-902.

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