ON A CLASS OF LIE ALGEBRAS

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In 1958, Block constructed a new class $\mathfrak{B}$ of simple Lie algebras, $\mathfrak{G}(G, \delta, f)$ [1]. Here $\mathfrak{G}$ is the direct sum of a finite number of finite elementary p-groups, $G_0, \cdots, G_n$, $p>2$ and $\delta=\delta_1+\cdots+\delta_n$ where δ_i is a nonzero element of G_i. Let F be a field of characteristic p. Then f is a nondegenerate skew-symmetric biadditive form defined on each G_i by $f_i(\alpha, \beta)=g_i(\alpha)h_i(\beta)-g_i(\beta)h_i(\alpha)$ for α, β in G_i and where g_i and h_i are additive functions on G_i to F with $g_i(\delta_i)=0$. To each $\alpha\neq 0, -\delta$ of G the formal symbol $v(\alpha)$ is assigned. Then $\mathfrak{L}(G, \delta, f)$ is the vector space over F with the $v(\alpha)$'s as a basis. The multiplication in $\mathfrak{L}(G, \delta, f)$ is defined by

$$v(\alpha)v(\beta) = \sum_{i=0}^{n} f_i(\alpha_i, \beta_i)v(\alpha + \beta - \delta_i)$$

where α_i and β_i are the components of α and β in G_i and where δ_0 and $v(0)$ denote 0.

Schafer [5] showed that each of these Lie algebras can be realized as the derived algebra of the algebra of inner derivations of a simple, nodal, Lie-admissible noncommutative Jordan algebra A. If $\mathfrak{L}$ is the class of all such Lie algebras that can be realized in this manner then Schafer's result is that $\mathfrak{L}\supseteq\mathfrak{B}$. The main result of this paper is to show that $\mathfrak{L}$ contains $\mathfrak{B}$ properly.

1. In this and subsequent sections A will denote a finite dimensional, simple, nodal, Lie-admissible, noncommutative Jordan algebra over a field F of characteristic $p>2$. Define A^+ to be the algebra that is the same vector space as A but has a product $x \circ y$ defined in terms of the product xy of A by

$$x \circ y = \tfrac{1}{2}(xy + yx).$$

Define A^- to be the algebra that is the same vector space as A but has a product $[x, y]$ defined in terms of the product xy by

$$[x, y] = xy - yx.$$

Kokoris [3] has shown that A^+ is the commutative, associative algebra $F[x_1, \cdots, x_n]$ of polynomials in $x_1, \cdots, x_n$ over F with the restriction that $x_i^p=0$. Hence $A^+=F1+N$ where N is the set of nilpotent elements of A^+.

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The multiplication in A can be given by

$$fg = f \circ g + \frac{1}{2} \sum \frac{\partial f}{\partial x_i} \circ \frac{\partial g}{\partial x_j} \circ c_{ij} \tag{1}$$

where $\partial/\partial x_i$ are the ordinary partial differential operators and $c_{ij} = x_i x_j - x_j x_i$. We shall confine our attention to the case $n=2$. In [5] it was shown that a pair of generators x and y could be chosen for A^+ such that

$$yx - xy = yD(x) = 1 + \alpha x^{p-1} \circ y^{p-1} \tag{2}$$

for some α in F. By the multiplication given in (1) we see that α completely determines the algebra A.

THEOREM 1. *If A_1 and A_2 are two algebras such that A_1^+ and A_2^+ have two generators and if they are defined by*

$$vD_1(x) = 1 + \alpha_1 x^{p-1} \cdot y^{p-1} \tag{3}$$

and

$$vD_2(u) = 1 + \alpha_2 u^{p-1} \cdot v^{p-1} \tag{4}$$

respectively then A_1 and A_2 are isomorphic if and only if $\alpha_1 = \alpha_2$.

PROOF. The sufficiency of the condition $\alpha_1 = \alpha_2$ is of course trivial. Therefore we shall assume A_1 and A_2 are isomorphic. In fact we can assume $A = A_1 = A_2$ and (x, y) and (u, v) are two pairs of generators for A^+. Then both u and v can be expressed as polynomials in x and y. Jacobson [2] has shown that any representatives in A^+ of the elements of any basis of the 2 dimensional space $N/N \circ N$ will serve as a pair of generators. Hence if x and y are generators of A^+ then so also are $x_1 = x + f(y)$ and $y_1 = y$ where $f(y)$ is of degree at least 1 in y. For if the cosets with representatives x and y form a basis for the space $N/N \circ N$ then so also do the cosets with representatives $x + \alpha y$ and y for any α in F. Clearly, this pair of generators will also satisfy (3) for the same α_1 since $y^i D_1(y) = 0$ and $x_1^{p-1} \circ y_1^{p-1} = x^{p-1} \circ y^{p-1}$. In the same manner, we can replace x_1 and y_1 by $x_2 = x_1$ and $y_2 = y_1 + g(x_1)$ and still retain (3).

If $u = \delta_1 x + \delta_2 y + h(x, y)$ and $v = \delta_3 x + \delta_4 y + q(x, y)$ where $h(x, y)$ and $q(x, y)$ are of degree at least 2 in x and y then, since $vD_2(u) = 1 + \alpha_2 u^{p-1} \circ v^{p-1}$, we have $\delta_1\delta_4 - \delta_2\delta_3 = 1$. Therefore either $\delta_1\delta_4 \neq 0$ or $\delta_2\delta_3 \neq 0$. Without loss of generality we can assume that the coefficient of x in u and the coefficient of y in v is not zero. For if $\delta_1\delta_4 = 0$ we can replace u and v by $u_1 = -v$ and $v_1 = u$ and still retain (2). Now by a suitable choice of the functions $f(y)$ and $g(x_1)$ above we can assume there is a pair of generators x_2, y_2 such that

$$y_2D_1(x_2) = 1 + \alpha_1 x_2^{p-1} \circ y_2^{p-1}$$

and

$$(5)\qquad \begin{aligned} u &= \delta_1 x_2 + h'(x_2, y_2) \circ x_2, \\ v &= \delta_1^{-1} y_2 + q'(x_2, y_2) \circ y_2, \end{aligned}$$

where h' and q' are of degree at least 1 in x_2 and y_2.

We shall assume x and y are such a pair of generators. We have $u^{p-1} \circ v^{p-1} = \delta_1^{p-1}\delta_1^{-p+1}x^{p-1} \circ y^{p-1} = x^{p-1} \circ y^{p-1}$. Write

$$u = \sum_{i,j=0}^{p-1} \alpha_{ij} x^i \circ y^j,$$

$$v = \sum_{i,j=0}^{p-1} \beta_{ij} x^i \circ y^j$$

and note that $\alpha_{0j} = \beta_{i0} = 0$. The coefficient of $x^{p-1} \circ y^{p-1}$ in the expression

$$\begin{aligned} vD_2(u) &= \sum_{i,j,s,t=0}^{p-1} (it - js)\alpha_{ij}\beta_{st} x^{i+s-1} \circ y^{j+t-1} \circ (1 + \alpha_1 x^{p-1} \circ y^{p-1}) \\ &= 1 + \alpha_2 u^{p-1} \circ v^{p-1} = 1 + \alpha_2 x^{p-1} \circ y^{p-1} \end{aligned}$$

will occur on the left only if either (a) $i+s-1=j+t-1=0$ or (b) $i+s-1=j+t-1=p-1$. If (a), then the coefficient is $(\alpha_{10}\beta_{01}-\alpha_{01}\beta_{10})\alpha_1$. But $\alpha_{10}=\delta_1=\beta_{01}^{-1}$ and $\beta_{10}=\alpha_{01}=0$. Therefore such a term will have a coefficient α_1. If (b), then $i \equiv -s$ and $j \equiv -t$ modulo p and $(it-js) \equiv 0$ modulo p. Hence we must have $\alpha_1=\alpha_2$ and the proof is complete.

2. Schafer [5] has shown that the algebra A associated with the algebra $\mathfrak{G}(G, \delta, f)$ (if A^+ has only two generators) has generators x and y such that either

$$(6)\qquad yD(x) = \beta(1 + x) \circ (1 + y)$$

or

$$(7)\qquad yD(x) = \beta$$

for some nonzero β in F. In the latter case by replacing x by $\beta^{-1}x$ we see that A is an algebra satisfying (2) with $\alpha = 0$. In the former case, Schafer has shown [5, p. 322] that A is an algebra that satisfies (2) with $\alpha = -\beta^{p-1}$. We let $\mathfrak{D}(A)$ be the set of inner derivations of A and $\mathfrak{D}'(A)$ the derived algebra of the algebra of inner derivations of

A. Clearly if A_1 and A_2 are isomorphic so will $\mathfrak{D}'(A_1)$ and $\mathfrak{D}'(A_2)$ be isomorphic. However, the following theorem shows that the converse does not hold for arbitrary fields F.

THEOREM 2. *If A_1 and A_2 are two algebras defined by the field elements α_1 and α_2 respectively then $\mathfrak{D}'(A_1)$ and $\mathfrak{D}'(A_2)$ are isomorphic if and only if there is a nonzero δ in F such that $\alpha_1=\delta^{p-1}\alpha_2$.*

PROOF. Assume x and y are generators of A_1^+ such that $yD_1(x) =1+\alpha_1 x^{p-1}\circ y^{p-1}$. We can assume that $\alpha_1\neq 0$. For if $\alpha_1=0$ then the dimension of $\mathfrak{D}'(A_1)$, and hence $\mathfrak{D}'(A_2)$, is p^2-2 [5, Theorem 6]. Therefore $\alpha_2=0$ and A_1 and A_2 are isomorphic. Now if $\alpha_1\neq 0$ then $\mathfrak{D}'(A_1)\cong\mathfrak{D}(A_1)\cong A_1^-/F1$ [5, p. 320].

Let σ be an isomorphism from $A_2^-/F1$ onto $A_2^-/F1$. Since each element of $A_i^-/F1$ is a coset of the ideal $F1$ of A_i^- and contains a unique element of N_i we can consider σ as a mapping of N_1 onto N_2. We let $\sigma(x^i\circ y^i)=z_{ij}$. (When convenient we shall use the symbol "$\equiv$" to indicate the congruence relation induced in A_i^- by the ideal $F1$).

LEMMA 1. *If $s, t\in A_i^-$ and $sD_i(t)\equiv 0$ then $sD_i(t)=0$.*

PROOF. If $sD_i(t)\equiv 0$ then there is a $\delta\in F$ such that $sD_i(t)=\delta$. Assume $\delta\neq 0$. Then s and t must be a pair of generators of A_i^+. But this implies that (7) is satisfied contradicting our assumption above.

We return to the proof of the theorem. Since the elements $x^i\circ y^j$, $0\leqq i, j\leqq p-1$ with not both i and j equal to zero, form a basis for the vector space N_1, the elements z_{ij} form a basis for the vector space N_2. Hence there must be a pair, say $u=z_{st}$ and $v=z_{mn}$, that are generators of A_2^+. Assume that both $\max(s, t)>1$ and $\max(m, n)>1$. Then

$$
(8)\qquad \begin{aligned}
(x^s\circ y^t)D_1(x^{p-1}\circ y^{p-1}) &= (s-t)x^{p+s-2}\circ y^{p+t-2}=0,\\
(x^m\circ y^n)D_1(x^{p-1}\circ y^{p-1}) &= (m-n)x^{p+m-2}\circ y^{p+n-2}=0.
\end{aligned}
$$

But since σ is an isomorphism on $A_1^-/F1$ we must have

$$uD_2(z_{p-1,p-1})\equiv vD_2(z_{p-1,p-1})\equiv 0.$$

By Lemma 1 we have

$$uD_2(z_{p-1,p-1}) = vD_2(z_{p-1,p-1}) = 0.$$

By (1) we see that $wD_2(z_{p-1,p-1})=0$ for all $w\in A_2$. It follows that $z_{p-1,p-1}=0$ [5, Lemma 2]. This is of course a contradiction of the definition of $z_{p-1,p-1}$. Hence we must have either $s, t\leqq 1$ or $m, n\leqq 1$. Say $s, t\leqq 1$. Again by (8) we can not have $s=t$. So assume $s=1$ and $t=0$ to get $\sigma(x)=u$.

Now using $z_{p-1,0}$ in the same way we used $z_{p-1,p-1}$ we see that we must have $m \leqq 1$. By direct computation we see that there are two types of terms that annihilate $x^m \circ y^n$ in $A_1^-/F1$. These are either of the form $w = x^{im} \circ y^{in}$ or $x^i \circ y^{p+1-j}$ for $j \leqq n$. For

$$(x^m \cdot y^n) D_1(x^i \cdot y^k) = (in - mk) x^{m+i-1} \cdot y^{n+k-1} = 0$$

if and only if either $in - mk = 0$ or $n+k-1 \geqq p$. No matter if $m=0$ or 1, the first possibility holds precisely for these terms of the form $x^{im} \circ y^{in}$ for any nonnegative integer i. Clearly, the second possibility holds precisely for those terms of the form $x^i \circ y^{p+1-j}$ where $j \leqq n$ and i is arbitrary. Hence if $n \geqq 2$ the subspace generated in A_1^- by such w's is of dimension $p(n-1)+r$ where r is the number of independent terms of the form $x^{im} \circ y^{in}$. Therefore the dimension of the subspace of elements in $A_2^-/F1$ that annihilate v must be $p(n-1)+r$. However, if $z = \sum \beta_{ij} u^i \circ v^j$ and $zD_2(v) \equiv 0$ then we must have $\sum i\beta_{ij} u^{i-1} \circ v^j \circ uD_2(v) = 0$. Since A_2 is simple $uD_2(v)$ must be nonsingular [4]. Therefore $\sum i\beta_{ij} u^{i-1} \circ v^j = 0$ and z is a polynomial in v. But the subspace generated in $A_2^-/F1$ by such z's is $p-1$. Therefore $n<2$ and $\sigma^{-1}(v)$ is either $x \circ y$ or y. Assume $\sigma^{-1}(v) = x \circ y$. Then $(x \circ y)D_1(x) = x$ so we must have $vD_2(u) \equiv u$ and $vD_2(u) = \delta + u$ for some $\delta \in F$. Since $vD_2(u)$ must be nonsingular we have $\delta \neq 0$ and $[v \circ (\delta+u)^{-1}]D_2(u) = 1$. But as argued above we see that such an assumption gives rise to a contradiction. Hence $\sigma^{-1}(v) = y$.

Recall that above we showed that the only polynomials that annihilate v were the polynomials in v. Hence it follows that $\sigma(y^i) = f_i(v)$ is a polynomial in v. Analogously, $\sigma(x^i) = g_i(u)$ is a polynomial in u. Conversely, by arguing on the dimension of the subspace generated by the powers of y we see that $\sigma^{-1}(v^i) = f_i'(y)$, a polynomial in y, and $\sigma^{-1}(u^i) = g_i'(x)$, a polynomial in x. We must have for $i>1$ that

$$\sigma(x^{i-1}) = \sigma(x^i D_1(y)) \equiv g_i(u) D_2(v) = \frac{\partial g_i(u)}{\partial u} \circ uD_2(v)$$

is a polynomial in u. Therefore, since u^2 is a linear combination of the g_i's we must have $u \circ uD_2(v)$ a polynomial in u also. But then $uD_2(v) = h_1(u) + u^{p-1} \circ h_2(v)$. Since a similar restriction holds for $vD_2(u)$ we must have

$$vD_2(u) = \delta_1 + \delta_2 u^{p-1} \circ v^{p-1}.$$

We shall now show by induction on the sum $i+j$ that

$$\sigma(x^i \circ y^j) = \delta_1^{-i-j+1} u^i \circ v^j. \tag{9}$$

Clearly, (9) holds if $i+j=1$. Also, if $i+j>1$ and $j>0$ then

$$\sigma(jx^i\circ y^{j-1}) = \sigma([x^i\circ y^j]D_1(x)) \equiv \sigma(x^i\circ y^j)D_2(u)$$
$$= \partial\sigma(x^i\circ y^j)/\partial v\circ vD_2(u) \equiv \delta_1^{-i-j+2}u^i\circ v^{j-1}.$$

Therefore $\delta_1\partial\sigma(x^i\circ y^j)/\partial v+\mu\delta_2u^{p-1}\circ v^{p-1}\equiv j\delta_1^{-i-j+1}u^i\circ v^{j-1}$ where μ is the coefficient of v in $\sigma(x^i\circ y^j)$. Since $\mu\delta_2u^{p-1}\circ v^{p-1}$ is the only term of degree $p-1$ in v we have $\mu=0$. It follows that $\sigma(x^i\circ y^j)=\delta_1^{-i-j+1}u^i\circ v^j\circ h(u)$. If $i=0$ then $\sigma(y^j)$ is a polynomial in v. Hence $h(u)$ is a constant β. If the constant β is nonzero then z_{ij} and u are generators of A_2^+. But this implies, repeating the argument presented in this proof that $\sigma^{-1}(z_{ij})=\epsilon y$ for some ϵ in F. Hence $z_{ij}=v$, $\beta=0$ and the induction holds if $i=0$. If $i\neq 0$ we can repeat the above argument using $\sigma(x^i\circ y^j)D_2(y)$ to get $\sigma(x^i\circ y^j)=\delta_1^{-i-j+1}u^i\circ v^j$. Therefore (9) holds for all i and j. However, $\alpha_1\sigma(x^{p-1}\circ y^{p-1})\equiv vD_2(y)\equiv\delta_2u^{p-1}\circ v^{p-1}$ since $yD_1(x)\equiv\alpha_1x^{p-1}\circ y^{p-1}$. But then $\alpha_1\delta_1^{-2p+3}=\delta_2$. Now replace the generators u and v in A_2^+ by $u'=\delta_1^{-1}u$ and $v'=v$ to get

$$v'D_2(u') = \delta_1^{-1}(vD_2(u)) = 1+\delta_1^{-1}\delta_2u^{p-1}\circ v^{p-1}$$
$$= 1+\alpha_1\delta_1^{-2p+2}u^{p-1}\circ v^{p-1} = 1+\alpha_1\delta_1^{-p+1}(\delta_1^{-1}u)^{p-1}\circ v^{p-1}$$
$$= 1+\alpha_1\delta_1^{-p+1}u'^{p-1}\circ v'^{p-1}.$$

Hence the necessity of the condition for an isomorphism holds.

Conversely, let x and y be generators of A_1^+ and u and v be generators of A_2^+ such that

$$yD_1(x) = 1+\alpha_1x^{p-1}\circ y^{p-1}$$
$$vD_2(u) = 1+\alpha_1\delta_1^{-p+1}u^{p-1}\circ v^{p-1}$$

for some nonzero $\delta_1\in F$. Define the linear mapping σ from A_1 to A_2 on the basal elements $x^i\circ y^j$ by

$$\sigma(x^i\circ y^j) = \delta_1^{-j+1}u^i\circ v^j.$$

A straightforward computation shows that σ is an isomorphism of $A_1^-/F1$ onto $A_2^-/F1$. As noted above each algebra of $\mathfrak{B}$ can be obtained from an algebra A satisfying (2) with either $\alpha=0$ or $\alpha=-\beta^{p-1}$ for some $\beta\in F$. It was shown [4, Theorem 3] that those algebras of $\mathfrak{B}$ with the corresponding $\alpha=0$ or $-\beta^{p-1}$ are of dimension p^2-2 and p^2-1 respectively. Therefore there are at least two nonisomorphic algebras in $\mathfrak{B}$. However from Theorem 2 we see that all of the algebras of $\mathfrak{B}$ obtained from an algebra A with $\alpha=-\beta^{p-1}$ are isomorphic. Hence

COROLLARY. *There are two nonisomorphic types of algebras in $\mathfrak{B}$ cor-*

responding to an A such that A^+ has two generators; one of dimension p^2-1 and one of dimension p^2-2.

To construct an algebra in $\mathfrak{L}$ but not in $\mathfrak{B}$ we need only choose a field F containing an element α such that $x^{p-1}+\alpha$ is irreducible over F.

COROLLARY. *The class $\mathfrak{B}$ is properly contained in in the class $\mathfrak{L}$.*

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