LOWER BOUNDS FOR SOLUTIONS OF HYPERBOLIC INEQUALITIES

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1. **Introduction.** Let D denote a bounded domain in E^n and I the interval $1 \leqq t < \infty$. Let L be the second-order hyperbolic operator

$$L = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial}{\partial x_j}\right) \tag{1.1}$$

defined on $R = D \times I$. Introducing the norms

$$\|u(t)\|_0^2 = \int_D u^2 \, dx,$$

$$\|u(t)\|_1^2 = \int_D \left[\left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2\right] dx,$$

for functions u in $C^2(R)$, Protter [4] investigated the asymptotic behavior of solutions of inequalities of the form

$$\|Lu(t)\|_0 \leqq \phi(t)\|u(t)\|_1. \tag{1.2}$$

If Γ is the boundary of D, he found that any solution of (1.2) which satisfies the conditions

$$u = 0 \quad \text{on} \quad \Gamma \times I,$$

$$\lim_{t \to \infty} t^\alpha \|u(t)\|_1 = 0 \quad \text{for all } \alpha > 0,$$

must vanish identically, provided that

$$\phi(t) = O(t^{-1}), \qquad \frac{\partial a_{ij}}{\partial t} = O(t^{-1}). \tag{1.3}$$

Conditions for other types of asymptotic behavior have also been studied by Protter [5].

It is the purpose of this paper to find sufficient conditions for the existence of lower bounds of the form

$$\|u(t)\|_1 \geqq C\|u(t_0)\|_1[K(t)]^{-1}, \qquad t \geqq t_0 \geqq 1,$$

where C is a positive constant and K is a differentiable function satisfying

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$$(1.4) \qquad K(t) > 0, \qquad K'(t) \geqq 0, \qquad \lim_{t\to\infty} K(t) = \infty.$$

In particular, it will be shown that in the case $K(t) = t^\alpha$, a lower bound exists under conditions somewhat weaker than (1.3). The results will also be extended to symmetric hyperbolic operators.

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2. **Second-order hyperbolic inequalities.** Let L be the operator defined by (1.1). We assume $a_{ij} = a_{ji} \in C^1(R)$, and suppose that there are positive constants m and M such that

$$m \sum_{i=1}^n \xi_i^2 \leqq \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \leqq M \sum_{i=1}^n \xi_i^2.$$

For functions $u \in C^2(R)$ we introduce the norm

$$\|u(t)\|^2 = \int_D \left[\left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right] dx,$$

which is equivalent to the norm $\|u(t)\|_1$. If $u=0$ on $\Gamma \times I$, it is easily seen that

$$\frac{d}{dt}\|u(t)\|^2 = 2\int_D \frac{\partial u}{\partial t} Lu dx + \int_D \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial t}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} dx.$$

Hence for any function K satisfying (1.4), we have the identity

$$(2.1) \qquad \begin{aligned} \frac{d}{dt}[K^2(t)\|u(t)\|^2] &= 2K(t)K'(t)\|u(t)\|^2 + 2K^2(t)\int_D \frac{\partial u}{\partial t} Lu dx \\ &\quad + K^2(t)\int_D \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial t}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} dx. \end{aligned}$$

Assume u is a solution of

$$(2.2) \qquad \|Lu(t)\|_0 \leqq \phi(t)\|u(t)\|,$$

such that $u=0$ on $\Gamma \times I$, and let ψ be a function satisfying

$$(2.3) \qquad \left|\int_D \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial t}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} dx\right| \leqq 2\psi(t)\|u(t)\|^2.$$

Applying Schwarz's inequality and (2.2) to the second term on the right-hand side of (2.1), and applying (2.3) to the third term of the same expression, we find that

$$\frac{d}{dt}\left[K^2(t)\|u(t)\|^2\right] \geqq 2K^2(t)\|u(t)\|^2\left[\frac{K'(t)}{K(t)} - f(t)\right],$$

where we have set $f=\phi+\psi$. It follows that

$$\frac{d}{dt}\log[K(t)\|u(t)\|] \geqq \frac{K'(t)}{K(t)} - f(t) \tag{2.4}$$

if $\|u(t)\| \neq 0$.

THEOREM. *Let u be a solution of* (2.2) *such that $u=0$ on $\Gamma\times I$. If $\|u(t_0)\| \neq 0$ and either*

(i) $(K'/K)^{1/p-1}f \in L_p(1, \infty)$ *for some p,* $1 \leqq p < \infty$,

or

(ii) $Kf/K' \in L_\infty(1, \infty)$ *and* $\|Kf/K'\|_\infty \leqq 1$,

then there exists a positive constant C such that

$$\|u(t)\| \geqq C\|u(t_0)\|[K(t)]^{-1}, \qquad t \geqq t_0 \geqq 1. \tag{2.5}$$

PROOF. We first assume that $\|u(t)\| \neq 0$ for $t \geqq t_0$. Integrating (2.4) between t_0 and t we obtain

$$\log\frac{K(t)\|u(t)\|}{K(t_0)\|u(t_0)\|} \geqq \log\frac{K(t)}{K(t_0)} - \int_{t_0}^t f\,ds. \tag{2.6}$$

In case (i), Hölder's inequality implies that

$$\left|\int_{t_0}^t f\,ds\right| \leqq \left[\int_{t_0}^t \left|\left(\frac{K'}{K}\right)^{-1/q} f\right|^p ds\right]^{1/p}\left[\int_{t_0}^t \frac{K'}{K}\,ds\right]^{1/q}$$
$$\leqq N\left[\log\frac{K(t)}{K(t_0)}\right]^{1/q},$$

where N is a constant and $1/p+1/q=1$. Hence, since $\lim_{t\to\infty} K(t)=\infty$, we see that the right-hand side of (2.6) is bounded below, and (2.5) follows. Under case (ii), the right-hand side of (2.6) is easily seen to be non-negative.

To prove that the assumption $\|u(t)\| \neq 0$ is valid for all $t \geqq t_0$, we suppose the contrary. Let $t_1 > t_0$ be the least value of t for which $\|u(t)\| = 0$. Then from the preceding result we find that (2.5) holds for $t_0 \leqq t < t_1$. By the continuity of the norm, we must have $\|u(t_1)\| \neq 0$. This completes the proof of the theorem.

If $K(t)=t^\alpha$, $\alpha>0$, conditions (i) and (ii) become: *either* $t^{1-1/p}f \in L_p(1, \infty)$ *for some* p, $1\leqq p<\infty$, *or* $tf\in L_\infty(1, \infty)$ *and* $\|tf\|_\infty\leqq\alpha$, which include Protter's conditions (1.3). For $K(t)=e^{\alpha t}$, $\alpha>0$, the conditions for the corresponding lower bound are: $f\in L_p(1, \infty)$ *for some* p, $1\leqq p<\infty$, *or* $f\in L_\infty(1, \infty)$ *and* $\|f\|_\infty\leqq\alpha$. These include the conditions obtained by Protter in [5], and are comparable to those found by Protter [3], Cohen and Lees [2] and Agmon and Nirenberg [1] for solutions of parabolic inequalities.

3. **Symmetric hyperbolic inequalities.** Let u be a k-component vector function in $C^1(R)$ and denote the components of u by u^j, $j=1, 2, \cdots, k$. For such functions we define

$$Lu = A_0\frac{\partial u}{\partial t} + \sum_{i=1}^{n} A_i\frac{\partial u}{\partial x_i},$$

where the A_i, $i=0, 1, \cdots, n$, are symmetric k-by-k matrices with elements in $C^1(R)$, and A_0 is positive definite. We take as norms the quantities

$$\|u(t)\|_0^2 = \int_D (u, u)dx,$$

$$\|u(t)\|^2 = \int_D (A_0u, u)dx,$$

with

$$(u, v) = \sum_{j=1}^{k} u^jv^j.$$

Since A_0 is symmetric, we have

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2 &= \int_D\left(2A_0\frac{\partial u}{\partial t} + \frac{\partial A_0}{\partial t}u, u\right)dx\\ &= \int_D\left(2Lu - 2\sum_{i=1}^{n} A_i\frac{\partial u}{\partial x_i} + \frac{\partial A_0}{\partial t}u, u\right)dx.\end{aligned}$$

Similarly, it follows that

$$\int_D\left(A_i\frac{\partial u}{\partial x_i}, u\right)dx = -\frac{1}{2}\int_D\left(\frac{\partial A_i}{\partial x_i}u, u\right)dx,$$

for functions u which vanish on the boundary $\Gamma\times I$. Thus, defining

$$B = \frac{\partial A_0}{\partial t} + \sum_{i=1}^{n} \frac{\partial A_i}{\partial x_i},$$

we find that

$$\frac{d}{dt}\|u(t)\|^2 = \int_D (2Lu + Bu, u)dx. \tag{3.1}$$

Suppose u is a solution of (2.2) and u vanishes on $\Gamma \times I$. Let ψ be a function satisfying

$$\left| \int_D (Bu, u)dx \right| \leqq 2\psi(t)\|u(t)\|^2.$$

Then the identity (3.1) implies that u satisfies the inequality (2.4), so the theorem of §2 is also valid in the present case.

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