THE DEGREE OF APPROXIMATION BY LINEAR OPERATORS

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I. Introduction. In 1959 Berman [1] observed the following: If \( T \) denotes the unit circle, \( C(T) \) the continuous functions on \( T \), and \( E_n(f) \) the error in the best approximation in the sup norm to \( f \) by a trigonometric polynomial of order \( n \), then there could not exist a sequence of linear operators \( T_n \) mapping \( C(T) \) into the trigonometric polynomials of order \( n \), which satisfied

\[
\|f - T_nf\| \leq KE_n(f)
\]

for some fixed constant \( K \) and \( f \in C(T) \). The reason for this is easy to see. Suppose (1) is satisfied for some sequence \( \{T_n\} \) and all \( f \in C(T) \). If \( \Pi_n \) denotes the space of trigonometric polynomials of order \( \leq n \), then for \( f \in \Pi_n, E_n(f) = 0 \). Hence, \( T_nf = f \) or \( T_{2n} = T_n \). But by a theorem of Nikolaev [3, p. 494], \( \|T_n\| \geq K \log n \). Since \( E_n(f) \to 0 \), this is a contradiction. A similar observation holds for \( L_1(T) \) where \( E_n(f) \) is the best approximation by a trigonometric polynomial of order \( n \) to \( f \) in the \( L_1 \) norm.

In this note I would like to elaborate on this observation of Berman's and make some applications to Fourier series which appear to be new. Let \( T_n \) be a sequence of bounded linear operators from \( C(T) \) into \( \Pi_n \). In place of \( E_n(f) \) let \( D_n(f) \) be any continuous mapping from \( C(T) \) to the non-negative reals which vanishes on \( \Pi_n \). In addition to the case \( D_n(f) = E_n(f) \) we may take \( D_n(f) = \|f - P_nf\| \) where \( P_n \) is a bounded projection of \( C(T) \) onto \( \Pi_n \). Then, if \( T_{2n} \neq T_n \) for each \( n \), it

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is never true that \( \|f - T_n f\| = O(D_n(f)) \) for all \( f \). This is still true even if \( \sup_n D_n(f) = \infty \) for some \( f \). Specifically the set of functions \( f \in C(T) \) for which there exist constants \( n_f, k_f \) depending on \( f \) for which
\[
\|f - T_n f\| \leq k_f D_n(f), \quad n \geq n_f,
\]
is of the first category in \( C(T) \). An interesting consequence of this lemma is obtained by taking \( D_n(f) = \|f - S_n f\|, \) \( S_n f \) the \( n \)th partial sum of the Fourier series and \( T_n f = \sigma_n f = (S_1 f + \cdots + S_n f)/n \). Then of course, \( \|f - \sigma_n f\| \to 0 \), and, therefore, the set of functions \( f \) such that no sequence \( S_n f \to f \) uniformly on \( T \) is a set of the first category in \( C(T) \). At first glance this appears mildly astonishing since those \( f \) for which \( \sup_n \|S_n f\| = \infty \) are of the second category. This set of the first category is not empty, however, since in 1944 Menshov \([2]\) constructed an example of a continuous function \( f \) such that for each sequence \( S_n f \) there exists a point \( x \in T \) such that \( (S_n f)(x) \) diverges at \( x \).

The observations concerning norm convergence hold verbatim in \( L_1 \). Also \( \{f \in L_1 : \lim \inf_n \|f - S_n f\|_1 > 0\} \) is not empty. For if \( \{a_n\} \) is convex, and \( a_n \to 0 \), then \( \frac{1}{2} a_0 + \sum_{n=1}^\infty a_n \cos nx \) defines a function in \( L_1 \) for which \( \|f - S_n f\|_1 = \pi a_n \|S_n f\| + o(1) \). Thus \( \lim_n \|S_n f\|_1 \) may be infinite. (See \([4, pp. 182-185]\) for details.)

II. The following elementary result is the key to these observations.

**Lemma.** Let \( E \) be a Banach space, \( \{E_n\} \) a sequence of closed subspaces satisfying \( E_n \subseteq E_{n+1} \) and \( \cap \cup E_n = E \). For each \( n \) let \( T_n \) be a bounded linear operator mapping \( E \) into \( E_n \) and let \( D_n \) be a continuous function mapping \( E \) into the non-negative reals which vanishes on \( E_n \). Let \( F_{k,n} = \{x : \|x - T_n x\| \leq k D_m(x), m \geq n\} \). Then \( T_{k,n} \) is a closed set, and if \( F_{k,n} \) contains an open set it follows that for some integer \( m_0 \), \( T_n = T_{m_0} \) for \( m \geq m_0 \).

**Proof.** In the applications we usually have either \( D_n(x) = \|x - P_n x\| \) where \( P_n \) is a projection of \( E \) onto \( E_n \) or \( D_n(x) = \inf_{y \in E_n} \|x - y\| \). Suppose there exists an \( x_0 \in E \), \( \delta > 0 \), such that \( \{x : \|x - x_0\| < \delta\} \subset F_{k,n} \). By the density of \( \cup_j E_j \) we may assume \( x_0 \in E_{m_0} \) for some \( m_0 \geq n \). For \( m \geq m_0 \), if \( x \in E_m \) and \( \|x\| < \delta \), then \( x + x_0 \in F_{k,n} \) and
\[
\|x + x_0 - T_m(x + x_0)\| \leq k D_m(x + x_0) = 0.
\]
Therefore, \( T_m x = x \), and by the homogeneity of \( T_m \), \( T_m x = x \) for all \( x \in E_m \), or \( T_n = T_m \).

**Corollary.** Under the same assumptions if \( T_n \neq T_m \) for each \( m \), then
$F = \{ x \in E : \text{there exist constants } n_x, k_x \text{ for which } \| x - T_m x \| \leq k_x D_m(x), \ m \geq n_x \}$ is of the first category in $E$.

**Proof.** $F = \bigcup F_{k,n}, \ k, n = 1, 2, \ldots$.

The observations concerning Fourier series are special cases of the following

**Corollary.** If $T_m x \to x$ for each $x$ and $T_m^k \neq T_m$ for each $m$ then \( \{ x : \lim \inf_m D_m(x) > 0 \} \) is of the first category in $E$.

The lemma, together with Nikolaev’s theorem and the uniform boundedness principle, yields a sharper form of Berman’s theorem.

**Corollary.** Let $T_m$ be bounded linear operators mapping $C(T)$ into $\Pi_m$. Then \( \{ f : \| f - T_m f \| \leq k E_m(f), \ m \geq n \} \) is of the first category in $C(T)$.

If $T_m$ is the Fejer operator $\sigma_m$, then there are nonconstant functions with this property. For by a classical theorem of Bernstein’s [3, p. 99] if $\alpha_m \downarrow 0$, there exists an $f \in C(T)$ such that $E_m(f) = \alpha_m$. But if $\alpha_m = 1/m^\alpha$, $0 < \alpha \leq 1$, then for such $f$,

$$\| \sigma_m f - f \| \leq \frac{k}{m^\alpha},$$

cf. [4, pp. 120–123].

**Bibliography**


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