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THE DEGREE OF APPROXIMATION BY LINEAR OPERATORS

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I. **Introduction.** In 1959 Berman [1] observed the following: If T denotes the unit circle, $C(T)$ the continuous functions on T, and $E_n(f)$ the error in the best approximation in the sup norm to f by a trigonometric polynomial of order n, then there could not exist a sequence of linear operators T_n mapping $C(T)$ into the trigonometric polynomials of order n, which satisfied

$$\|f - T_n f\| \leqq K E_n(f) \tag{1}$$

for some fixed constant K and $f \in C(T)$. The reason for this is easy to see. Suppose (1) is satisfied for some sequence $\{T_n\}$ and all $f \in C(T)$. If Π_n denotes the space of trigonometric polynomials of order $\leqq n$, then for $f \in \Pi_n$, $E_n(f) = 0$. Hence, $T_n f = f$ or $T_n^2 = T_n$. But by a theorem of Nikolaev [3, p. 494], $\|T_n\| \geqq K \log n$. Since $E_n(f) \to 0$, this is a contradiction. A similar observation holds for $L_1(T)$ where $E_n(f)$ is the best approximation by a trigonometric polynomial of order n to f in the L_1 norm.

In this note I would like to elaborate on this observation of Berman's and make some applications to Fourier series which appear to be new. Let T_n be a sequence of bounded linear operators from $C(T)$ into Π_n. In place of $E_n(f)$ let $D_n(f)$ be any continuous mapping from $C(T)$ to the non-negative reals which vanishes on Π_n. In addition to the case $D_n(f) = E_n(f)$ we may take $D_n(f) = \|f - P_n f\|$ where P_n is a bounded projection of $C(T)$ onto Π_n. Then, if $T_n^2 \neq T_n$ for each n, it

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is never true that $\|f-T_nf\|=O(D_n(f))$ for all f. This is still true even if $\sup_n D_n(f)=\infty$ for some f. Specifically the set of functions $f\in C(T)$ for which there exist constants n_f, k_f depending on f for which

$$\|f - T_nf\| \leqq k_fD_n(f), \qquad n \geqq n_f,$$

is of the first category in $C(T)$. An interesting consequence of this lemma is obtained by taking $D_n(f)=\|f-S_nf\|$, S_nf the nth partial sum of the Fourier series and $T_nf=\sigma_nf=(S_1f+ \cdots +S_nf)/n$. Then of course, $\|f-\sigma_nf\|\to 0$, and, therefore, the set of functions f such that no sequence $S_{n_k}f\to f$ uniformly on T is a set of the first category in $C(T)$. At first glance this appears mildly astonishing since those f for which $\sup_n \|S_nf\|=\infty$ are of the second category. This set of the first category is not empty, however, since in 1944 Menshov [2] constructed an example of a continuous function f such that for each sequence $S_{n_k}f$ there exists a point $x\in T$ such that $(S_{n_k}f)(x)$ diverges at x.

The observations concerning norm convergence hold verbatim in L_1. Also $\{f\in L_1\colon \lim\inf_n\|f-S_nf\|_1>0\}$ is not empty. For if $\{a_n\}$ is convex, and $a_n\to 0$, then $\frac{1}{2}a_0+\sum_{n=1}^{\infty} a_n \cos nx$ defines a function in L_1 for which $\|f-S_nf\|_1=\pi a_n\|S_n\|+o(1)$. Thus $\lim_n\|S_nf\|_1$ may be infinite. (See [4, pp. 182–185] for details.)

II. The following elementary result is the key to these observations.

Lemma. *Let E be a Banach space, $\{E_n\}$ a sequence of closed subspaces satisfying $E_n\subset E_{n+1}$ and $\text{Cl}\bigcup E_n=E$. For each n let T_n be a bounded linear operator mapping E into E_n and let D_n be a continuous function mapping E into the non-negative reals which vanishes on E_n. Let $F_{k,n}\equiv\{x\colon \|x-T_mx\|\leqq kD_m(x),\ m\geqq n\}$. Then $T_{k,n}$ is a closed set, and if $F_{k,n}$, contains an open set it follows that for some integer m_0, $T_m^2=T_m$ for $m\geqq m_0$.*

Proof. In the applications we usually have either $D_n(x)=\|x-P_nx\|$ where P_n is a projection of E onto E_n or $D_n(x)=\inf_{y\in E_n}\|x-y\|$. Suppose there exists an $x_0\in E$, $\delta>0$, such that $\{x\colon \|x-x_0\|<\delta\}\subset F_{k,n}$. By the density of $\bigcup_j E_j$ we may assume $x_0\in E_{m_0}$ for some $m_0\geqq n$. For $m\geqq m_0$, if $x\in E_m$ and $\|x\|<\delta$, then $x+x_0\in F_{k,n}$ and

$$\|x + x_0 - T_m(x + x_0)\| \leqq kD_m(x + x_0) = 0.$$

Therefore, $T_mx=x$, and by the homogeneity of T_m, $T_mx=x$ for all $x\in E_m$, or $T_m^2=T_m$.

Corollary. *Under the same assumptions if $T_m^2\neq T_m$ for each m, then*

$F=\{x\in E$: there exist constants n_x, k_x for which $\|x-T_mx\|\leqq k_xD_m(x)$, $m\geqq n_x\}$ is of the first category in E.

PROOF. $F=\bigcup F_{k,n}$, k, $n=1, 2, \cdots$.

The observations concerning Fourier series are special cases of the following

COROLLARY. *If $T_mx\to x$ for each x and $T_m^2\neq T_m$ for each m then $\{x: \liminf_m D_m(x)>0\}$ is of the first category in E.*

The lemma, together with Nikolaev's theorem and the uniform boundedness principle, yields a sharper form of Berman's theorem.

COROLLARY. *Let T_m be bounded linear operators mapping $C(T)$ into Π_m. Then $\{f: \|f-T_mf\|\leqq k_fE_m(f),\ m\geqq n_f\}$ is of the first category in $C(T)$.*

If T_m is the Fejer operator σ_m, then there are nonconstant functions with this property. For by a classical theorem of Bernstein's [**3**, p. 99] if $\alpha_m\downarrow 0$, there exists an $f\in C(T)$ such that $E_m(f)=\alpha_m$. But if $\alpha_m=1/m^\alpha$, $0<\alpha\leqq 1$, then for such f,

$$\|\sigma_mf-f\|\leqq\frac{k}{m^\alpha},$$

cf. [**4**, pp. 120–123].

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