A THEOREM ON HOMOTOPICALLY EQUIVALENT $(2k+1)$-MANIFOLDS[1]

YUEN-FAT WONG

1. **Introduction.** In this note we shall prove a theorem on homotopically equivalent manifolds of odd dimension. The theorem is similar to one of Novikov's theorems [**6**, Theorem 2]. We kill the excess homotopy subgroup of middle dimension of the manifold considered below by removing a handle body.

2. **Preliminaries.** Manifolds considered here are compact, oriented, connected and differentiable. A homotopy sphere is a closed manifold which is of the same homotopy type as a sphere. If we are given disjoint differentiable imbeddings $\phi_i: \partial D_i^k \times D_i^k \to \partial D^{2k}$, the boundary of D^{2k}, where D^{2k} is a $2k$-cell and D^k is a k-cell, then a handle body considered here is a manifold obtained from $D^{2k} \cup \bigcup_{i=1}^s D_i^k \times D_i^k$, by identifying $\partial D_i \times D_i$ with its image in ∂D^{2k}, with corners smoothed.

THEOREM. *Let M_1^{2k-1} and M_2^{2k-1} be two simply connected, homtopically equivalent manifolds with $k \geqq 3$, satisfying the following hypotheses:*

(i) *There is a simply connected manifold N^{2k} with boundary $\partial N = M_2^{2k-1} \cup (-M_1^{2k-1})$ and the relative homotopy groups.*

$$\Pi_q(N, M_1) = 0 \qquad \text{for } q = 1, 2, 3, \cdots, (k-1).$$

(ii) *There is a continuous map $g: N^{2k} \to M_2^{2k-1}$ such that $g \mid M_2^{2k-1}$ is the identity and $g \mid M_1^{2k-1}$ is the homotopy equivalence $f: M_1 \to M_2$.*

Then there exists a homotopy sphere Σ which bounds a handle body such that M_1^{2k-1} is diffeomorphic to $M_2^{2k-1} \# \Sigma$, where $\#$ stands for connected sum.

PROOF. Let us consider the following maps:

$$M_2^{2k-1} \xrightarrow{i_2} N^{2k} \xrightarrow{g} M_2^{2k-1},$$

where i_2 is the inclusion map and g is given by the hypotheses. Since $g \circ i_2$ is the identity, g induces inverse maps in the homology and homotopy sequences. Therefore we have the following splitting exact sequences.

Received by the editors September 12, 1964.

[1] This paper is a portion of the author's doctoral thesis. The author would like to express his sincere appreciation to his thesis advisor, Professor W. Browder.

$$\cdots \longrightarrow \Pi_i(M_2) \underset{\nwarrow \quad /}{\overset{i_{2\#}}{\longrightarrow}} \Pi_i(N) \longrightarrow \Pi_i(N, M_2) \longrightarrow \cdots$$
$$g_\#$$
$$\cdots \longrightarrow H_i(M_2) \underset{\nwarrow \quad /}{\overset{i_{2*}}{\longrightarrow}} H_i(N) \longrightarrow H_i(N, M_2) \longrightarrow \cdots$$
$$g_*$$
$$\cdots \longleftarrow H^i(M_2) \underset{\backslash \quad \nearrow}{\overset{i_2^*}{\longleftarrow}} H^i(N) \longleftarrow H^i(N, M_2) \longleftarrow \cdots$$
$$g^*$$

Since the composition $g \circ i_1$ is equal to f, that is,

$$M_1 \overset{i_1}{\longrightarrow} N \overset{g}{\longrightarrow} M_2$$
$$\nwarrow \quad f' \quad /$$

and $f \circ f' \sim id_{M_2}, f' \circ f \sim id_{M_1}$ we have the identity $= (f' \circ f)_* = (f' \circ g \circ i_1)_* = (f' \circ g)_* \circ i_{1*}$. The corresponding exact sequences of (N, M_1) also split. By the relative Hurewicz theorem, the first nonvanishing relative homology group is isomorphic to the homotopy group, that is, $\Pi_k(N, M_2) \cong H_k(N, M_2)$. By the relative Poincaré duality and the universal coefficient theorem, this group is free abelian, that is,

$$H_k(N, M_2) \cong H^k(N, M_1) \cong \operatorname{Hom}(H_k(N, M_1), Z).$$

It is of finite rank r, because of compactness. Now we may identify $H_k(N, M_2)$ with $H_k(N, M_1)$ and consider the intersection pairing: $H_k(N, M_2) \otimes H_k(N, M_1) \rightarrow Z$. The quadratic form of this pairing has a unimodular matrix of coefficients [**5**]. By virtue of the Hurewicz theorem and the splitting homotopy exact sequences we can regard $H_k(N, M_2)$ as a direct summand of $\Pi_k(N)$. We realize a set of generators of $H_k(N, M_2)$ by imbedded spheres S_i^k [**4**, p. 50]. We can assume that these spheres intersect each other transversally and only at isolated points. To construct a handle body from these spheres we proceed as follows:

(1) For each sphere S_i^k, we construct in S_i^k a path γ_i such that it passes through the points of intersection once and only once. For example, we can order the points $p_1, p_2, \cdots, p_t$, in S_i^k. Draw a path starting with p_1 to p_2 without crossing other points, p_2 to p_3, etc. finally to p_t. The path is constructed to be nonself-intersecting and not closed. In N, γ_i may intersect γ_j, $i \neq j$. We call the union of these paths K^1. We have therefore a 1-complex K^1 in N.

(2) Let U_i be a cell neighborhood of γ_i in S_i^k. We can choose U_i suitably such that if $i \neq j$ then $(S_i - U_i) \cap (S_j - U_j) = \varnothing$. We can

choose a $2k$-cell V_i which is a neighborhood of $(S_i - U_i)$ in N such that if $i \neq j$, then $V_i \cap V_j = \emptyset$.

(3) It is easy to see that $N - (V_1 \cup V_2 \cup \cdots \cup V_r)$ is simply connected. Recall that N is simply connected and there are only finite number of V's.

(4) By a lemma of Penrose-Whitehead-Zeeman [7, Lemma 2.7], there exists a $2k$-cell E containing the 1-complex K^1. Recall that K is the union of $\{\gamma_i\}$. This cell is in $N - (V_1 \cup V_2 \cup \cdots \cup V_r)$. Take a small neighborhood of each imbedded sphere. These neighborhoods together with the $2k$-cell E form a handle body. We may smooth some combinatorial elements by a theorem of [2].

We have thus realized a handle body in N. The intersection pairing has a property that the quadratic form of the pairing has a unimodular matrix of coefficients. By a lemma of Wall [9, p. 169] the boundary of the handle body is a homotopy sphere. We remove the interior of the handle body and a solid tube connecting the handle body and M_2. As a result we have a new manifold N' with $\partial N' = -M_1 \cup M_2 \# \Sigma$, where Σ is the homotopy sphere bounding the handle body. We claim that M_1 and $M_2 \# \Sigma$ are h-cobordant. It is sufficient to show the following:

(1) M_1, M_2 and N' are simply connected. We have only to show that N' is simply connected. Let X_1 be N' and X_2 be the handle body and the solid tube, then $X_1 \cap X_2$ is a part of the homotopy sphere and a cylinder (in fact $X_1 \cap X_2$ is contractible), and $X_1 \cup X_2 = N$. Applying a theorem of van Kampen [3] to $X_1 \cup X_2$ we see that N' is simply connected.

(2) To show that M_1 and $M_2 \# \Sigma$ are deformation retracts on N', it is sufficient to show that the inclusion map induces an isomorphism in homology, that is, to show that $H_q(N', M_2 \# \Sigma; Z) = 0$ for all q. First let us examine the following Mayer-Vietoris sequence:

$$\cdots \to H_j(X_1 \cap X_2) \to H_j(X_2) \oplus H_j(X_1) \xrightarrow{\lambda} H_j(X_1 \cup X_2) \to \cdots.$$

Note that $X_1 \cap X_2$ is contractible. Therefore λ is an isomorphism. We have

$$H_i(N') \cong H_i(X_1) \cong H_i(X_1 \cup X_2) \qquad \text{for } i \neq k, \text{ and}$$

$$H_k(N') \cong H_k(N)/\mathrm{Ker}\, g_*.$$

Secondly, let us cover each sphere S_i^k by a $(k+1)$-cell. Then the handle body together with these cells, that is, $X_2 \cup$cells, is a contractible piece. From the sequence

$$\cdots \to H_j(M_2) \to H_j(N' \cup X_2 \cup \text{cells}) \to H_j(N' \cup X_2 \cup \text{cells}, M_2) \to \cdots.$$

we have $H_j(M_2) \cong H_j(N' \cup X_2 \cup \text{cells})$ (isomorphic) and

$$H_j(N', \cup X_2 \cup \text{cells}, M_2) = 0.$$

To simplify the notation we denote $X_2 \cup$cells by H^+. Now it is easy to see that $(N' \cup H^+, M_2)$ and $(N', M_2 \# \Sigma)$ are homotopically equivalent to $(N'/\Sigma, M_2 \# \Sigma/\Sigma)$. Therefore,

$$H_*(N', M_2 \# \Sigma) = H_*(N'/\Sigma, M_2 \# \Sigma/\Sigma),$$
$$0 = H_*(N' \cup H^+, M_2) = H_*(N'/\Sigma, M_2 \# \Sigma/\Sigma)$$
$$H_*(N', M_2 \# \Sigma) = 0,$$

so that $H_*(M_2 \# \Sigma) \cong H_*(N')$.

By a theorem of J. H. C. Whitehead [**10**], $M_2 \# \Sigma$ is a deformation retract of N'. Similarly M_1 is a deformation retract on N'. Hence M_1 and $M_2 \# \Sigma$ are h-cobordant. The h-cobordism theorem of Smale [**8**] implies that they are diffeomorphic.

References

1. W. Browder, *Homotopy type of differentiable manifolds*, (Mimeographed notes) Aarhus Universitet, Denmark, 1962.

2. M. Hirsch, *On combinatorial submanifolds of a differentiable manifold*, Comment. Math. Helv. **37** (1962), 103–110.

3. P. J. Hilton and S. Wylie, *Homology theory*, Cambridge Univ. Press, New York, 1960.

4. J. Milnor, *A procedure for killing the homotopy groups of differentiable manifolds*, Proc. Sympos. Pure Math., Vol. 3, American Mathematical Society, Providence, R. I., 1961; pp. 39–55.

5. J. Milnor, *On simply connected* 4-*manifolds*, Internat. Sympos. Algebraic Topology, pp. 122–128. Universidad National Autonoma de Mexico and UNESCO, Mexico City, 1958.

6. S. P. Novikov *Diffeomorphisms of simply connected manifolds*, Dokl. Akad. Nauk **143** (1962), 11–12=Soviet Math Dokl. **3** (1962), 540–543.

7. R. Penrose, J. H. C. Whitehead and E. C. Zeeman, *Imbedding of manifolds in Euclidean space*, Ann. of Math. **73** (1961), 613–622.

8. S. Smale, *On the structure of manifolds*, Amer. J. Math. **84** (1962), 384–399.

9. C. T. C. Wall, *Classification of* $(n-1)$-*connected* $2n$-*manifolds*, Ann. of Math. **75** (1962), 163–189.

10. J. H. C. Whitehead, *Combinatorial homotopy*. I, Bull. Amer. Math. Soc. **55** (1949), 213–245.

Cornell University and
De Paul University