NONDIFFERENTIABILITY OF RETRACTIONS OF C^n TO SUBVARIETIES

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Theorem 5.2 of [1] says that if x is a singular point of a complex subvariety V of an open subset of C^n, then for no open neighborhood N of x in C^n is there a holomorphic retraction of N to $V \cap N$. It will be proved here that there isn't even a real-C^1 retraction.

LEMMA. *If a complex subvariety V of an open subset U of C^n is a real-C^1 submanifold, it is a complex submanifold.*

PROOF. The set $R(V)$ of points of V having open neighborhoods in V which are complex submanifolds of open subsets of C^n is dense in V. For $y \in V$ let V_y denote the tangent space at y to the real-C^1 submanifold V of U. V_y is a real-linear subspace of C^n for all $y \in V$. Let A be the set of all $y \in V$ such that V_y is a *complex*-linear subspace of C^n. Then A is a relatively closed subset of V. Also, since $R(V) \subseteq A$, A is dense in V. Hence $A = V$. V_y is a complex-linear subspace of C^n for all $y \in V$. So V is a complex submanifold of U.

THEOREM. *Let V be a complex subvariety of an open set $U \subseteq C^n$. Let $x \in V$. Suppose there exist an open neighborhood U_1 of x in U and a real-C^1 retraction F of U_1 to $V \cap U_1$. Then $x \in R(V)$.*

PROOF. Choose a real-C^1 submanifold S of an open neighborhood U_2 of x in U_1 such that $S \supseteq V \cap U_2$ and the real dimension s of S is minimal for this property. If $s=0$ then x is an isolated point of V, and hence in $R(V)$. So suppose $s>0$. If f is a real-C^1 function on S such that $(df)_x \neq 0$, then the set S_1 of zeros of f is a submanifold of real dimension $s-1$ in some neighborhood of x in S, so by the minimality of the dimension of S, S_1 contains no neighborhood of x in V, so f vanishes on no neighborhood of x in V. In other words, if f is a real-C^1 function on S vanishing on $V \cap U_2$, then $(df)_x = 0$. If f and g are real-C^1 functions on S which coincide on $V \cap U_2$, then $(df)_x = (dg)_x$. Since F equals the identity on $V \cap U_2$, $(d(F|S))_x = (dI)_x$, where I is the identity on S. Thus $(d(F|S))_x$ has rank s. By the inverse function theorem $F(S) \subseteq V$ is a neighborhood of x in S. There is an open neighborhood U_3 of x in U_2 such that $V \cap U_3 = S \cap U_3$. By the lemma, $V \cap U_3$ is a complex submanifold of U_3. $x \in R(V)$.

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ON PIECEWISE LINEAR IMMERSIONS

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The purpose of this note is to prove an existence theorem for immersions of piecewise linear manifolds in Euclidean space. A more comprehensive theory of piecewise linear immersions has been worked out by Haefliger and Poenaru [**1**].

All maps, manifolds, microbundles, etc. are piecewise linear unless the contrary is explicitly indicated.

Let M be a manifold without boundary, of dimension n. Denote the tangent microbundle of M by τ_M, and the trivial microbundle over M of (fibre) dimension k by ϵ^k. Let

$$\nu: M \xrightarrow{i} E \xrightarrow{j} M$$

be a microbundle of dimension k such that E is a manifold. An *immersion* of M in R^{n+k} is a locally one-one map $f: M \to R^{n+k}$.

I say f has a *normal bundle of type* ν if there is an immersion $g: E \to R^{n+k}$ such that $gi=f$. (It is unknown whether f necessarily has a normal bundle, or whether all normal bundles of f are of the same type.)

The converse of the following theorem is trivial.

THEOREM. *Assume that if $k=0$, then M has no compact component. There exists an immersion of M in R^{n+k} having a normal bundle of type ν if there exists an isomorphism*

$$\phi: \tau_m \oplus \nu \to \epsilon^{n+k}$$

PROOF. We may assume that $i(M)$ is a deformation retract of the total space E of ν. By Milnor [**3**], $\tau_E | i(M)$ is isomorphic to $\tau_M \oplus \nu$; it follows from the existence of ϕ that τ_E is trivial. According to [**3**]

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