A NOTE ON INDEFINITE INTEGRALS

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In this note all functions are measurable, complex valued and periodic on the real line; all distributions are tempered and periodic on the real line. Periodic means periodic of period 2π. If f is a function $\|f(x)\|_{p,d\mu(x)}$ is the L^p norm of f, taken with respect to the measure $d\mu$ on $[0, 2\pi]$; $\|f\|_p$ is the L^p norm taken with respect to ordinary Lebesgue measure; and L^p is the class of those functions f such that $\|f\|_p < \infty$. If F is a function, we let $\Delta^2 F(x, h) = F(x+h) - 2F(x) + F(x+h)$.

M. Weiss and A. Zygmund have shown (Theorem 1 and Theorem 3 in [3]):

THEOREM A. (i) *If* $\|\Delta^2 F(x, h)\|_{p,dx} = O[|h|/|\log|h||]^\beta$, $1 \leqq p \leqq 2$, $\beta > 1/p$, *then F is equal, a.e., to the indefinite integral of a function f, and* $f \in L^p$. (ii) *If* $\|\Delta^2 F(x, h)\|_{p,dx} = O[|h|/|\log|h||]^\beta$, $2 \leqq p < \infty$, $\beta > 1/2$, *then F is equal, a.e., to the indefinite integral of a function f, and* $f \in L^p$. (iii) *If* $\|\Delta^2 F(x, h)\|_{\infty,dx} = O[|h|/|\log|h||]^\beta$, $\beta > 1/2$, *then F is equal, a.e., to the indefinite integral of a function f and $f \in L^p$ for all p,* $1 \leqq p < \infty$.

REMARK. The assertion (iii) is, of course, an immediate corollary of (ii).

Our main objective is to supply a bound on the L^p norm of the derivative f, as well as indicating the connection of these results with a large body of results relating smoothness and differentiability conditions.

DEFINITION 1. Suppose $0 < \alpha < 2$, $1 \leqq p$, $q \leqq \infty$. If $F \in L^p$ and $M_{\alpha;p,q} = \| \|\Delta^2 F(x, h)\|_{p,dx}/|h|^\alpha\|_{q,dh/|h|} < \infty$, we say $F \in \Lambda_\alpha^{pq}$ and set $\|F\|_{\alpha;p,q} = M_{\alpha;p,q} + \|F\|_p$.

Since $\|F\|_{\alpha;p,q} \leqq \|F\|_{\alpha;\infty,q}$, Theorem A is seen to be a corollary of the following result.

THEOREM. *Suppose* $1 \leqq p < \infty$, $r = \min[p, 2]$, $F \in \Lambda_1^{pr}$. *Then F is equal, a.e., to the indefinite integral of a function f belonging to L^p, and there is a constant C_p, depending only on p, such that* $\|f\|_p \leqq C_p \|F\|_{1;p,r}$.

PROOF. We need to show that if $F(x) \sim \sum c_n e^{inx}$, then $\sum inc_n e^{inx}$ is the Fourier series of a function f in L^p and that $\|f\|_p \leqq C_p \|F\|_{1;p,r}$.

We state several definitions and lemmas, from which the result easily follows. (The crucial step is Lemma 2.)

Received by the editors July 9, 1964.

DEFINITION 2. If f is a distribution, $f(x)\sim\sum d_n e^{inx}$, $J^\alpha f$ (α complex), is the distribution $J^\alpha f(x)\sim\sum (1+|n|^2)^{-\alpha/2} d_n e^{inx}$.

This is the Bessel potential operator of order α for the one-dimensional periodic case, and is discussed in the paragraphs preceding Lemma 19 in [1].

LEMMA 1. *For α complex, $\{b_n^\alpha\}_{-\infty}^{\infty}$, $b_n^\alpha=(|n|^2/(1+|n|^2))^\alpha$, $n\neq 0$; $b_0^\alpha=0$. are the Fourier coefficients of a finite Borel measure.*

This is Lemma 19 of [1]. It is proved there for α real, but the proof easily extends to complex α. (We use only the case $\alpha=1/2$.)

DEFINITION 3. If $f=J^\alpha\phi$, $\phi\in L^p$, α complex, $1\leqq p\leqq\infty$, we say $f\in L^p$ and define $\|f\|_{p,\alpha}=\|\phi\|_p$.

LEMMA 2. *If $1\leqq p<\infty$, $r=\min[p, 2]$ $(0<\alpha<2)$ then $\Lambda_\alpha^{pq}\subset L_\alpha^p$. The inclusion map is continuous.*

The spaces defined in Definition 3 and Definition 1 are discussed in detail in Chapter VIII of [1]. Lemma 2 is part of Theorem 15′ of [1], specialized to the one-dimensional case.

Let $f\to\tilde f$ be the conjugate mapping. That is if $f(x)\sim\sum d_n e^{inx}$, $\tilde f(x)\sim\sum i \operatorname{sgn} n\, d_n e^{inx}$.

LEMMA 3. *The conjugate operator maps Λ_α^{pq} continuously into itself, $0<\alpha<2$, $1\leqq p, q\leqq\infty$.*

This is the one-dimensional version of Theorem 3 of [2].

We proceed with the proof. We have $F(x)\in\Lambda_1^{pr}$, $F(x)\sim\sum c_n e^{inx}$. Lemma 3 asserts that $\sum i \operatorname{sgn} n c_n e^{inx}$ is the Fourier series of a function $g\in\Lambda^{pr}$ and that

$$\|g\|_{1;p,r}\leqq A_p\|F\|_{1;p,r} \tag{1}$$

for some A_p independent of F. Lemma 2 asserts that $g\in L_1^p$, so that (using Definition 1 and Definition 3) $\sum (1+|n|^2)^{1/2} i \operatorname{sgn} n c_n e^{inx}$ is the Fourier series of a function $h\in L^p$ and

$$\|h\|_p=\|J^{-1}g\|_p=\|g\|_{p,1}\leqq B_p\|g\|_{1;p,r} \tag{2}$$

for some B_p independent of F.

Lemma 1 now asserts that

$$\sum(|n|/(1+|n|^2)^{1/2})(1+|n|^2)^{1/2}\, i \operatorname{sgn} n c_n e^{inx} = \sum |n|\, i \operatorname{sgn} n c_n e^{inx}=\sum i n c_n e^{inx}$$

is the Fourier series of a function $f\in L^p$ and that (D_p independent of F),

$$\|f\|_p \leqq D_p\|h\|_p \leqq D_pB_p\|g\|_{1;p,r} \leqq D_pB_pA_p\|F\|_{1;p,r}. \tag{3}$$

The result follows with $C_p = D_pB_pA_p$. Q.E.D.

References

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3. M. Weiss and A. Zygmund, *A note on smooth functions*, Nederl. Akad. Wetensch. Proc. Ser. A **62** (1959), 52–58.

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