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PRIME RINGS WITH MAXIMAL ANNIHILATOR AND MAXIMAL COMPLEMENT RIGHT IDEALS[1]

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1. **Introduction.** Let R be a prime ring with a maximal annihilator right ideal and a maximal complement right ideal. Then there is a division ring D such that either R is isomorphic to a right order in the complete ring of linear transformations of a finite dimensional D-space, or for each positive integer n there is a subring $R^{(n)}$ of R which is isomorphic to a right order in the complete ring of linear transformations of an n-dimensional D-space. This is related to a result of N. Jacobson [2, p. 33] and extends a theorem of A. W. Goldie [1; Theorem 4.4] that a prime ring with maximum conditions on annihilator right ideals and complement right ideals is a right order in a simple ring with minimum condition on right ideals. R is also isomorphic to a weakly transitive ring of linear transformations of a vector space. This is a generalization of a theorem of R. E. Johnson [4; 3.3].

2. We assume throughout that R is a prime ring. The notation R_r^Δ (R_l^Δ) is used to denote the right (left) *singular ideal* of R, and L_r^* (L_l^*) is the lattice of closed right (left) ideals of R. An R-module is *uniform* if each pair of nonzero submodules has nonzero intersection. A right (left) ideal of R is *uniform* if it is uniform as right (left) R-module. For other definitions and notation see [6].

THEOREM 1. *R contains a maximal annihilator right ideal and a maximal complement right ideal if and only if $R_r^\Delta = (0)$ and L_r^* is atomic.*

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PROOF. Assume first that R contains a maximal annihilator right ideal x^r, $x\in R$, $x\neq 0$. Suppose $R_r^\Delta\neq(0)$. Then $R_r^\Delta x\neq(0)$; so there exists $y\in R_r^\Delta$ such that $yx\neq 0$. Now $xR\neq(0)$, so $y^r\cap xR\neq(0)$. If $xr\in y^r\cap xR$, $xr\neq 0$, then $r\in(yx)^r$ and $x^r\subset(yx)^r$, $x^r\neq(yx)^r$, a contradiction. Hence $R_r^\Delta=(0)$. If R contains a maximal complement C, then a complement A of C is an atom of L_r^*. If $B\in L_r^*$, then $BA\neq(0)$ and $bA\neq(0)$ for some $b\in B$. bA is an atom of L_r^* contained in B. Hence L_r^* is atomic.

Conversely, assume that $R_r^\Delta=(0)$ and L_r^* is atomic. Let A be an atom of L_r^* and $a\in A$, $a\neq 0$. Then a^r is a maximal annihilator right ideal and a maximal complement.

LEMMA. *Let V be a uniform quasi-injective R-module with zero singular submodule and let $D=\mathrm{Hom}_R(V, V)$. If $v\in V$, then Dv has nonzero intersection with each nonzero submodule if and only if there is a nonzero element $r\in R$ such that $r\cdot v^r=(0)$.*

PROOF. Let $v^{rl}=\{r\in R: r\cdot v^r=(0)\}$ and suppose $v^{rl}\neq(0)$. Let V_1 be a nonzero submodule of V. Let $w\in V_1$, $w\neq 0$. Then $w\cdot v^{rl}\neq(0)$. Let $r\in v^{rl}$, $wr\neq 0$. Then $v^r\subseteqq r^r\subseteqq(wr)^r$. By [6, 2.2], v and wr generate the same 1-dimensional D-subspace of V. Hence $Dv\cap V_1\neq(0)$.

Conversely, suppose that Dv has nonzero intersection with each nonzero submodule of V. There is a nonzero right ideal I of R such that $v^r\cap I=(0)$. $vI\cap Dv\neq(0)$. Let $i\in I$, $d\in D$ be such that $vi=dv\neq 0$. Then $v^r=(dv)^r=(vi)^r=i^r$. Thus $i\in v^{rl}$ and $v^{rl}\neq(0)$.

THEOREM 2. *Let R be a prime ring for which $R_r^\Delta=(0)$ and L_r^* is atomic. Then there is a division ring D such that* (1) *R is a right order in the ring of all linear transformations of a finite dimensional D-space, or* (2) *for each positive integer n there is a subring $R^{(n)}$ of R such that $R^{(n)}$ is a right order in the ring of all linear transformations of an n-dimensional D-space.*

PROOF. The maximal right quotient ring R' of R is primitive with minimal right ideal I'. We first show that I' is a quasi-injective R-module.[2] Let U be a nonzero R-submodule of I' and suppose α is a nonzero homomorphism of U into I'. Let $u\in U$ such that $\alpha(u)\neq 0$. $R_r'^\Delta=(0)$ because $R_r^\Delta=(0)$. Thus there is a nonzero right ideal J' of R' such that $\{r'\in R': ur'=0\}\cap J'=(0)$. Define $\tilde{\alpha}$ on $I'=uJ'$ by: $\tilde{\alpha}(ur')=\alpha(u)r'$, $r'\in J'$. $\tilde{\alpha}\in\mathrm{Hom}_R(I', I')$. Suppose $v\in U$, $v=ur'$, $r'\in J'$. Then $K=\{r\in R: r'r\in R\}$ is a large right ideal of R. If $r\in K$, then $\alpha(v)r=\alpha(vr)=\alpha(ur'r)=\alpha(u)r'r$; hence $r\in(\alpha(v)-\alpha(u)r')^r$. Since

[2] This fact is a consequence of the following unpublished result of R. E. Johnson: If $R_r^\Delta=(0)$ and $R\subset S\subset R'$, and if M is a quasi-injective S-module, then M is a quasi-injective R-module.

K is large and $R_r^\Delta=(0)$, we conclude $\alpha(v)=\alpha(u)r'=\bar{\alpha}(ur')=\bar{\alpha}(v)$. This proves that $\bar{\alpha}$ is an extension of α, and we conclude that I' is a quasi-injective R-module.

Now, let $I=I'\cap R$. I is a uniform R-module with zero singular submodule. Let $D=\mathrm{Hom}_R(I', I')=\mathrm{Hom}_{R'}(I', I')$. Then D is a division ring and $V = DI$ is a minimal quasi-injective extension of I. $D=\mathrm{Hom}_R(V, V)$. Let B be a basis for V as (left) vector space over D. If B is finite, then L_r^* is finite dimensional [6, 3.1]. Hence, in this case, R is a right order in a simple ring with minimum condition on right ideals [1, 4.4].

Suppose B is not finite and let n be a positive integer. Let V_1 be the subspace of V spanned by a subset $\{v_i\}_{i=1}^n$ of B. Choose $e\in R'$ such that $ve=v$, each $v\in V_1$, and such that e has rank n as linear transformation of V. Let $R^{(n)}=\{r\in R\cap eR' \colon V_1r\subseteq V_1\}$. $R^{(n)}$ is a subring of R isomorphic to a subring of the full ring Q of linear transformations of V_1. We show that $R^{(n)}$ is a right order in Q. Let $q\in Q$ of rank $k\geqq 1$. Let $\{w_i\}_{i=1}^n$ be a basis for V_1, and if $k<n$ we assume that this basis is chosen so that $\{w_i\}_{i=k+1}^n$ is a basis for the null space of q. Let $\{u_i\}_{i=1}^n$ be a basis for V_1 chosen so that $u_i=w_iq$, $1\leqq i\leqq k$. If $J_i=\bigcap_{j\neq i}w_j^r$ and $K_i=\bigcap_{j\neq i}u_j^r$, then $w_iJ_i\neq(0)$ and $u_iK_i\neq(0)$, $1\leqq i\leqq n$, by [6, 2.2]. Let $r_i\in J_i$ and $s_i\in K_i$ such that $w_ir_i\neq 0$, $u_is_i\neq 0$, $1\leqq i\leqq n$. Then $T=\{r\in R\colon er_ir\in R,\ es_ir\in R,\ 1\leqq i\leqq n\}$ is a large right ideal of R. By the Lemma and the fact that $R_r^\Delta=(0)$, $w_ir_iT\cap u_is_iT\cap Dv_i\neq(0)$. Let $a_i\in T$, $b_i\in T$ such that $w_ir_ia_i=u_is_ib_i$ is a nonzero element of Dv_i. If $r=\sum_{j=1}^k r_ja_j$ and $s=\sum_{j=1}^n s_jb_j$, then er and es are in $R^{(n)}$ and $w_iqes=w_ier$, $1\leqq i\leqq n$. Moreover, es is nonsingular on V_1. This proves that $R^{(n)}$ is a right order in Q.

COROLLARY. *R contains a subring which is a right order in a division ring.*

THEOREM 3. *Let R be a prime ring such that $R_r^\Delta=(0)$ and L_r^* is atomic. Let U be an atom of L_r^* and V the minimal quasi-injective extension of U as right R-module. If $D=\mathrm{Hom}_R(V, V)$ there exists a subring K of R which is a right order in D and such that if $\{v_i\}_{i=1}^n$ is a finite D-linearly independent subset of U and $\{y_i\}_{i=1}^n$ is a sequence in U, then there exists $r\in R$ and $k\in K$, $k\neq 0$, such that $v_ir=ky_i$, $1\leqq i\leqq n$.*

PROOF. Let R', I', and V be chosen as in the proof of Theorem 2. There exists $u\in U$ such that $u^r\cap I'=(0)$. Then $uI'=I'$. Let $e\in I'$ be such that $ue=u$. Then $e^2=e$ and $eR'e$ is a division ring isomorphic to D. Let $L=R'e\cap R$, and let $K=eR'e\cap R=(eR'\cap R)\cap(R'e\cap R)=U\cap L$. L is a left ideal of R and K is a subring of R and $eR'e$. Let

$d=er'e$, $r'\in R'$, $d\neq 0$. Then $dU\cap U\neq(0)$. Let $I=\{u\in U: du\in U\}$. Then $I\neq(0)$ and $I\cap L\neq(0)$. If $k_1\in I\cap L$, $k_1\neq 0$, then $dk_1\in K$. This proves that K is a right order in $eR'e$.

Let $\{v_i\}_{i=1}^n$ be a finite D-linearly independent subset of U and $\{y_i\}_{i=1}^n$ a sequence in U. For $1\leqq i\leqq n$, let $J_i=\bigcap_{j\neq i} v_j^r$. By [6, 2.2] $v_iJ_i\neq(0)$. Let $\bar{k}\in K$, $\bar{k}\neq 0$. Then $v_iJ_i\bar{k}\neq(0)$. Select $b_i\in J_i$ such that $v_ib_i\bar{k}\neq 0$. Then $v_ib_i\bar{k}\in K$ and $\bigcap_{i=1}^n (v_ib_i\bar{k})K\neq(0)$. Hence $v_ib_i\bar{k}k_i=k\neq 0$, $1\leqq i\leqq n$, for some k_i, k in K. If $r=\sum_{i=1}^n b_i\bar{k}k_iy_i$, then $v_ir=ky_i$, $1\leqq i\leqq n$. The theorem is proved.

Suppose now that we also assume in Theorem 3 that $R_l^\Delta=(0)$ and L_l^* is atomic. Then we can show that K, (and hence L) is also left uniform. For assume K_1 and K_2 are nonzero left ideals of K and that $K_1\cap K_2=(0)$. If $k\in K$, $k\neq 0$, then $K_1k\cap K_2k=(0)$. But $(Lk)^*$ is an atom of L_l^* by [4, 1.1]. If $0\neq k_1\in K_1$ and $0\neq k_2\in K_2$, then $Lk_1k\cap Lk_2k\neq(0)$. Suppose $l_1k_1k=l_2k_2k\neq 0$. Let $k'\in U$ such that $k'l_1k_1k=k'l_2k_2k\neq 0$. Then $k'l_1k_1\in K_1$ and $k'l_2k_2\in K_2$, and so $K_1k\cap K_2k\neq(0)$, a contradiction. Clearly L is uniform. Since K is left uniform it is a left order in D. This implies that a K-linearly independent subset of U is also D-linearly independent. This shows that [4, 3.3] is a consequence of Theorem 3.

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