INNER AND OUTER FUNCTIONS ON RIEMANN SURFACES

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1. Introduction. In this paper we generalize to Riemann surfaces the factorization theory for functions in the Hardy classes, \(H^p\), on the unit disk.

Let \(R\) be a region on a Riemann surface with boundary \(\Gamma\) consisting of a finite number of simple closed analytic curves such that \(R \cup \Gamma\) is compact and \(R\) lies on one side of \(\Gamma\). For \(1 \leq p < \infty\) \(H^p(R)\) is the class of functions \(F\) analytic on \(R\) such that \(|F|^p\) has a harmonic majorant. \(H^\infty(R)\) is the class of bounded analytic functions on \(R\). The above classes are the usual generalizations of the Hardy classes on the disk (cf. [4], [5], and [6]). However, to obtain a factorization of these functions which closely parallels the factorization on the disk we are led to more general classes of functions. To this end, we say that a (multiple-valued) analytic function \(F\) on \(R\) is multiplicative if \(|F|\) is single-valued and define \(MH^p(R), 1 \leq p < \infty\), to be the class of multiplicative analytic functions on \(R\) such that \(|F|^p\) has a harmonic majorant. Also, we define \(MH^\infty(R)\) to be the class of bounded multiplicative functions on \(R\).

Let \(d\mu\) be the harmonic measure on \(\Gamma\) with respect to some fixed point \(t_0 \in R\). If \(F \in MH^p\) then \(|F|\) has nontangential boundary values \(|F^*|\) a.e. \([d\mu]\) on \(\Gamma\). Moreover, \(|F^*| \in L^p(\Gamma, d\mu)\) and \(\log |F^*| \in L^1(\Gamma, d\mu)\) if \(F \neq 0\). These facts follow easily from the corresponding results on the disk. (Cf. [10, p. 496].)

We say \(F \in MH^p(R)\) is an outer function if

\[
\log |F(t_0)| = \int_{\Gamma} \log |F^*| \, d\mu.
\]

\(\Phi \in MH^\infty(R)\) is an inner function if \(|\Phi^*| = 1\) a.e. on \(\Gamma\). A nonvanishing inner function is said to be a singular function. An inner function, \(B\), is said to be a Blaschke product if

\[
|B(t)| = \exp \left[ -\sum_k p_k G(t, t_k) \right]
\]

for all \(t \in R\), where \(G\) is the Green’s function for \(R\), \(\{t_k\}\) is a sequence of points on \(R\), and \(\{p_k\}\) is a sequence of non-negative integers. When

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$R$ is the unit disk, the definitions above are equivalent to the classical ones.

**Theorem 1.** Let $F \in MH^p(R)$. Then $|F| = |B||S||F_i|$ where $B$ is a Blaschke product, $S$ is a singular function, and $F_i$ is an outer function in $MH^p(R)$. These factors are unique up to multiplicative constants of modulus one.

Since $H^p(R) \subset MH^p(R)$, this theorem subsumes a factorization of functions in $H^p(R)$. However, the factors of a single-valued function in $H^p(R)$ need not be single-valued.

Theorem 1 is stated in terms of the moduli of functions since there is no natural way to define the product of multiplicative functions; that is, the product would depend on the branches chosen.

In our proof we shall make use of the factorization on the disk. A convenient reference for this is [3].

It should be remarked that results related to ours are in [7] for $R$ an annulus and in [2] for the general case.

2. **Proof of Theorem 1.** Let $K = \{z|z| < 1\}$ and $T: K \rightarrow R$ be a universal covering map of $R$. Let $Q = \{q\}$ be the group of fractional linear transformations such that $T \circ q = T$. A function $f$, analytic on $K$, is said to be modulus invariant if $|f \circ q| = |f|$ for all $q \in Q$. It is easy to see that $f$ is modulus invariant on $K$ if, and only if, $f \circ T^{-1}$ is multiplicative on $R$. Also, for $f$ modulus invariant, $f \in H^p(K)$ if, and only if, $f \circ T^{-1} \in MH^p(R)$. For $F \in MH^p(R)$ and $f \circ T^{-1} = F$ let $f = bsf_i$ where $b$ is a Blaschke product, $s$ is a singular function, and $f_i$ is an outer function in $H^p(K)$. Then $f$ is modulus invariant and by Lemma 4.6 in [10] $bs$ and $f_i$ are also modulus invariant.

**Lemma.** $b$ is modulus invariant.

**Proof.** First observe that if $z$ is a zero of $f$ then $q(z)$ is a zero of $f$ with the same multiplicity for each $q \in Q$. Thus $b \circ q/b \in H^\infty(K)$ for each $q \in Q$ since $b$ and $b \circ q$ have the same zeros with the same multiplicities. Then for all $q \in Q$ $b/b \circ q = (b \circ q^{-1} \circ q)/(b \circ q) \in H^\infty(K)$. It follows that $|b| = |b \circ q|$ for all $q \in Q$.

Let $\Delta$ be the fundamental domain of $Q$ and $E$ the union of the free sides of $\Delta$; that is, $E = \Delta \cap \{|z| = 1\}$. Then the harmonic measure $d\mu$ corresponds to the measure

$$dm(\theta) = \frac{1}{2\pi} \left[ \sum_{q \in Q} \frac{1 - |q(z_0) - e^{i\theta}|^2}{|e^{i\theta} - q(z_0)|^2} \right] d\theta$$

on $E$ where $z_0 \in \Delta$ and $T(z_0) = t_0$. (Cf. [9, pp. 526, 529].) Moreover, $dm(\theta)$ and $d\theta$ are mutually absolutely continuous.
Let \( B = b \circ T^{-1}, \ S = s \circ T^{-1}, \) and \( F_1 = f_1 \circ T^{-1}. \) Then \( |F| = |B| |S| |F_1|. \) Since \( b \) is an inner function and \( s \) is a singular function, it is immediate by virtue of the relation between \( d\theta \) and \( dm(\theta) \) that \( B \) is an inner function and \( S \) is a singular function. (Cf. [10, Lemma 4.4].) It remains to show that \( B \) is a Blaschke product and \( F_1 \) is an outer function.

We shall consider \( F_1 \) first. Since \( dm(\theta) \) corresponds to \( d\mu, \) it follows that

\[
\int_{\Gamma} \log |F_1^*| \, d\mu = \int_{\Gamma} \log |f_1^*| \, dm(\theta).
\]

Now since the set \( \bigcup_{q \in Q} q(E) \) is of full measure on \( |z| = 1 \) (see [9, p. 525]) and \( |f_1^* \circ q| = |f_1^*| \) a.e. on \( |z| = 1, \) a change of variable for the integral on the right (see [9, pp. 526–528]) yields

\[
\int_{\Gamma} \log |F_1^*| \, d\mu = \int_{\Gamma} \log |f_1^*| \, dm(\theta)
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \log |f_1^*| \, \frac{1 - |z_0|^2}{|e^{i\theta} - z_0|} \, d\theta
\]

\[
= \log |f_1(z_0)| = \log |F_1(t_0)|.
\]

Therefore \( F_1 \) is an outer function.

Next we show that \( B \) is a Blaschke product. Let \( \{t_k\} \) be the set of zeros of \( F \) and \( p_k \) be the multiplicity of the zero at \( t_k. \) For each \( t_k \) let \( z_k \) be a point in \( K \) such that \( T(z_k) = t_k. \) Then for \( q \in Q, \) \( b \) has a zero at \( q(z_k) \) of multiplicity \( p_k. \) All the zeros of \( b \) occur in this way. Thus,

\[
|b(z)| = \prod_{k \in Q} \prod_{q \in Q} \left| \frac{q(z_k) - z}{1 - q(z_k)z} \right|^{p_k}.
\]

Now for \( z, z' \in K \)

\[
G(T(z), T(z')) = \sum_{q \in Q} \log \left| \frac{1 - q(z')z}{z - q(z')} \right|.
\]

(See [9, p. 529].) Hence for \( t = T(z) \)

\[
|B(t)| = |b(z)| = \prod_k \exp\left[-p_k G(t, t_k)\right] = \exp\left[-\sum_k p_k G(t, t_k)\right]
\]

Thus \( B \) is a Blaschke product and \( F = |B||S||F_1| \) is the desired factorization. Since \( b, s, \) and \( f_1 \) are unique up to multiplicative constants of modulus one, the same is true for \( B, S, \) and \( F_1. \) This completes the proof of Theorem 1.
3. Closed invariant subspaces of $H^2(R)$. Let $A(R)$ be the class of (single-valued) functions continuous on $R$ and analytic on $R$. A closed subspace $V$ of $H^2(R)$ is said to be invariant if $FV \subseteq V$ for all $F \in A(R)$. For $\Phi$ an inner function on $R$ let $V(\Phi) = \{F \in H^2(R) \mid |F|^2/|\Phi|^2\}$ has a harmonic majorant on $R$. In [10, Theorem 8.11] it is shown that $V(\Phi)$ is a closed invariant subspace of $H^2(R)$. The following theorem, which reduces to a well-known result of Beurling's [1, Theorem IV, p. 253] for the case $R = K$, is proved in [10, Theorem 2].

**Theorem 2.** If $V$ is a closed invariant subspace of $H^2(R)$ then there is an inner function $\Phi$ such that $V = V(\Phi)$.

By Theorem 1 we have for $F \in H^2(R)$, $|F| = |\Phi| |F_1|$ where $\Phi$ is an inner function. We call $\Phi$ an inner factor of $F$.

**Theorem 3.** Let $F \in H^2(R)$ and $V[F]$ be the smallest closed invariant subspace of $H^2(R)$ which contains $F$. Then $V[F] = V(\Phi)$ where $\Phi$ is an inner factor of $F$.

**Proof.** Clearly $V(\Phi) \supseteq V[F]$. By Theorem 2, $V[F] = V(\Phi_0)$ for some inner function $\Phi_0$. Let $H \in V(\Phi)$. We must show $H \in V(\Phi_0)$. For $f = F/T$ and $h = H \circ T$ let $f = \phi f_1$ and $h = \psi h_1$ be the inner-outer factorizations of $f$ and $h$ respectively such that $\Phi = \phi \circ T^{-1}$. Then $\Psi = \psi \circ T^{-1}$ is an inner factor of $H$. Let $\phi_0$ be a modulus invariant inner function such that $\Phi_0 = \phi_0 \circ T^{-1}$. Since $F \in V(\Phi_0)$, $|F|^2/|\Phi_0|^2$ has a harmonic majorant; and it follows that $f/\phi_0 \in H^2(K)$. This implies $\phi/\phi_0$ is an inner function. By a similar argument $\psi/\phi$ is an inner function. Thus $\psi/\phi_0 = (\psi/\phi)(\phi/\phi_0)$ is an inner function. Hence $|\Psi|/|\Phi_0|$ is bounded. This implies $H \in V(\Phi_0)$.

**Corollary 1.** Let $F \in H^2(R)$. Then $V[F] = H^2(R)$ if, and only if, $F$ is an outer function. (Cf. [7, Theorem 1, p. 128].)

The following result was proved by D. Sarason ([7, pp. 112, 128] and [8, Theorem 4, p. 596]). We offer a different proof.

**Corollary 2.** Let $R = \{z \mid r < |z| < 1\}$ and suppose $F \in H^2(R)$. Then $V[F] = H^2(R)$ if, and only if, for $0 \leq \delta \leq 1$

\[
(*) \quad \int_0^{2\pi} \log |F(re^{it})| \, dt = (1 - \delta) \int_0^{2\pi} \log |F^*(e^{it})| \, dt + \delta \int_0^{2\pi} \log |F^*(re^{it})| \, dt.
\]

**Proof.** By virtue of Corollary 1 it is sufficient to show that $F$ satisfies $(*)$ if, and only if, $F$ is an outer function. Suppose $F$ is an outer function, then
\[
\int_0^{2\pi} \log |F(r^*e^{it})| \, dt = \int_0^{2\pi} \frac{1}{2\pi} \int_{\partial R} \log |F^*(w)| \frac{\partial G(w, r^te^{it})}{\partial n} \, dw \, dt
\]

\[
= \int_{|w|=1} \log |F^*(w)| \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial G(w, r^te^{it})}{\partial n} \, dt \right| \, dw
\]

\[
+ \int_{|w|=-r} \log |F^*(w)| \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial G(w, r^te^{it})}{\partial n} \, dt \right| \, dw
\]

\[
= I_1(\delta) + I_2(\delta).
\]

Now \( \partial G(e^{it}, r^te^{it})/\partial n = \partial G(e^{-it}, r^te^{-it})/\partial n \). Thus for \( |w|=1 \)

\[
\int_0^{2\pi} \frac{\partial G(w, r^te^{it})}{\partial n} \, dt = \int_0^{2\pi} \frac{\partial G(e^{it}, r^t\bar{w})}{\partial n} \, dt
\]

= harmonic measure of \( \{ |z|=1 \} \) at \( r^t\bar{w} \)

\[
= 1 - (\log r/\log r) = 1 - \delta.
\]

Thus \( I_1(\delta) = (1-\delta) \int_0^{2\pi} \log |F^*(e^{it})| \, dt \) . A similar argument shows \( I_2(\delta) = \delta \int_0^{2\pi} \log |F^*(r^te^{it})| \, dt \) . Hence \( F \) satisfies (*)

To complete the proof we show that if \( F \) is not an outer function then (*) is not satisfied. Let \( F_1 \) be an outer factor of \( F \). Then \( |F| < |F_1| \) on \( R \) and \( |F_1| = |F_1| \) a.e. on \( \partial R \). We have shown that \( F_1 \)
satisfies (*). Thus (*) does not hold for \( F \).

REFERENCES

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