FOURIER COSINE TRANSFORMS WHOSE REAL PARTS ARE NON-NEGATIVE IN A STRIP

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1. **Introduction.** In a recent note [1] we obtained the Poisson integral representation of every function $u(x, y)$ which is positive and harmonic in a strip, $-1<y<1$. We here make use of this result to characterize those Fourier cosine transforms

$$\int_0^\infty \cos(x+iy)t\,\phi(t)\,dt$$

whose real parts are positive and integrable on $-\infty<x<\infty$ for each y in $-1<y<1$. The characterizing condition on $\phi(t)$ is that $\phi(t)\cosh t$ (defined for $t<0$ so as to be even) should be real and positive definite. As an example, we have the classical equations

$$\frac{1}{z^2+1}=\int_0^\infty \cos zr\,e^{-r}\,dr, \qquad z=x+iy,$$

$$u(x, y)=\operatorname{Re}\frac{1}{z^2+1}=\frac{x^2-y^2+1}{(x^2-y^2+1)^2+4x^2y^2}>0, \quad |y|<1,$$

$$\int_{-\infty}^\infty u(x, y)\,dx=\pi, \qquad |y|<1, \tag{1}$$

$$e^{-|r|}\cosh r=\frac{1}{2}\int_{-\infty}^\infty e^{irt}\,d\left[U(t)+\frac{1}{\pi}\tan^{-1}\frac{t}{2}\right]. \tag{2}$$

Here $U(t)$ is zero for $t<0$ and unity for $t>0$, so that the integrator function in (2) is increasing and bounded. Thus $e^{-|r|}\cosh r$ is positive definite in confirmation of the theory. Equation (1) can be checked directly or will follow from Corollary 2 below.

An analogous result for the sine-transform is also obtained.

2. **Positive integrable harmonic functions.** In [1] the Poisson integral representation of functions positive and harmonic in a strip was obtained. If such functions are also integrable over the whole doubly-infinite lines of the strip they also have a simple Fourier integral representation, which we now obtain. We use H and L to denote the classes of harmonic and integrable (on the whole x-axis) functions,

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respectively. Let us state Theorem 4 of [1], altered in notation only. Set

$$Q(x, y) = \frac{1}{4} \frac{\cos \frac{\pi}{2} y}{\cosh \frac{\pi}{2} x + \sin \frac{\pi y}{2}} \tag{3}$$

$$= \frac{1}{\pi} \frac{d}{dx} \tan^{-1} \left\{ \frac{\sin \frac{\pi y}{2} + e^{\pi x/2}}{\cos \frac{\pi y}{2}} \right\}.$$

THEOREM A. *A necessary and sufficient condition that $u(x, y)$ should be non-negative and harmonic in the strip $-1<y<1$ is that*

$$u(x, y) = [Ae^{\pi x/2} + Be^{-\pi x/2}] \sin \frac{\pi y}{2} + \int_{-\infty}^{\infty} Q(x - t, y)\, d\alpha(t) \tag{4}$$

$$+ \int_{-\infty}^{\infty} Q(x - t, -y)\, d\beta(t), \qquad -1 < y < 1,$$

where $A \geqq 0$, $B \geqq 0$, $\alpha(t)$ and $\beta(t)$ are nondecreasing.

To obtain our basic result we observe that $Q(x, y)$ is itself a positive definite function of x. This results from the fact that it is the Fourier transform of a positive function,

$$Q(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixt} \frac{\sinh (1 - y)t}{\sinh 2t}\, dt, \qquad -1 < y < 1. \tag{5}$$

See p. 36 of [2].

THEOREM 1. *A necessary and sufficient condition that $u(x, y) \in H$, $\in L$, $\geqq 0$ for $-1<y<1$ is that*

$$u(x, y) = \int_{-\infty}^{\infty} e^{ixr} \frac{\sinh (1 - y)r}{\sinh 2r} g(r)\, dr + \int_{-\infty}^{\infty} e^{ixr} \frac{\sinh (1 + y)r}{\sinh 2r} h(r)\, dr, \tag{6}$$

where $g(r)$ and $h(r)$ are positive definite.

To prove this it will be sufficient to show that under the added condition

$$\int_{-\infty}^{\infty} u(x, y)\, dx < \infty, \qquad -1 < y < 1 \tag{7}$$

the representations (4) and (6) are identical. Since every term of (4) is nonnegative it is clear that the above inequality cannot hold unless $A=B=0$. From (3) we see that

$$\int_{-\infty}^{\infty} Q(x,y)\,dx = \frac{1-y}{2}, \qquad -1 < y < 1.$$

Hence by Fubini's theorem (7) can then hold if and only if the nondecreasing functions α and β are also bounded. Then (7) takes the explicit form

$$\int_{-\infty}^{\infty} u(x,y)\,dx = \frac{1-y}{2}\int_{-\infty}^{\infty} d\alpha(t) + \frac{1+y}{2}\int_{-\infty}^{\infty} d\beta(t) < \infty.$$

Now substituting (5) in (4) we obtain

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha(t)\int_{-\infty}^{\infty} e^{i(x-t)r}\,\frac{\sinh(1-y)r}{\sinh 2r}\,dr$$
$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} d\beta(t)\int_{-\infty}^{\infty} e^{i(x-t)r}\,\frac{\sinh(1+y)r}{\sinh 2r}\,dr.$$

In view of the boundedness of α and β we may again apply Fubini's theorem to invert the order of integration, obtaining (6) with

$$g(r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itr}\,d\alpha(t),$$
$$h(r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itr}\,d\beta(t).$$

By Bochner's theorem [3] g and h are positive definite. This concludes the proof.

COROLLARY 1. *If to the conditions of the theorem is added that $u(x,-y)=u(x,y)$ they become necessary and sufficient that*

$$u(x,y) = \int_{-\infty}^{\infty} e^{ixr}\,\frac{\cosh yr}{\cosh r}\,g(r)\,dr, \qquad -1 < y < 1, \tag{8}$$

where $g(r)$ is positive definite.

For, from (6)

$$u(x,y) = \frac{u(x,y)+u(x,-y)}{2} = \int_{-\infty}^{\infty} e^{ixr}\,\frac{\cosh yr}{\cosh r}\,\frac{g(r)+h(r)}{2}\,dr.$$

The proof is concluded by an obvious change in notation.

A simple change of variable shows that if the strip $-1<y<1$ of Theorem 1 is replaced by $0<y<c$, then (6) becomes

$$(9)\quad u(x,y)=\int_{-\infty}^{\infty}e^{ixr}\frac{\sinh(c-y)r}{\sinh cr}g(r)dr+\int_{-\infty}^{\infty}e^{ixr}\frac{\sinh yr}{\sinh cr}h(r)dr,$$

where g and h are positive definite.

3. The Fourier cosine transform. The precise statement of the result of §1 follows.

THEOREM 2. *The conditions*

A. $u(x,y)\in H, \geqq 0, \in L(-\infty<x<\infty), -1<y<1,$

B. $u(-x,y)=u(x,-y)=u(x,y)$

are necessary and sufficient that

$$u(x,y)=\operatorname{Re}\int_{0}^{\infty}\cos(x+iy)t\,\phi(t)\,dt,\qquad -1<y<1,$$

where $\phi(|t|)\cosh t$ *is real and positive definite.*

Under Conditions A and B, Corollary 1 shows that

$$u(x,y)=\int_{-\infty}^{\infty}e^{ixr}\frac{\cosh yr}{\cosh r}g(r)\,dr,\qquad -1<y<1,$$

for some positive definite function $g(r)$. Denote the real and imaginary parts of g by g_1 and g_2 respectively. Then g_1 is even and positive definite; g_2 is odd. By B, $u(x, y)$ is even in x, so that

$$(10)\quad \begin{aligned} u(x,y)&=\int_{-\infty}^{\infty}\cos xr\frac{\cosh yr}{\cosh r}g_1(r)\,dr\\ &=\operatorname{Re}\int_{0}^{\infty}\cos(x+iy)r\,\phi(r)\,dr,\\ \phi(|r|)&=2g_1(r)/\cosh r,\qquad -\infty<r<\infty. \end{aligned}$$

This proves the sufficiency of Conditions A and B. The necessity follows easily.

COROLLARY 2. *Under the Conditions* A *and* B

$$\int_{-\infty}^{\infty}u(x,y)dx=\pi\phi(0),\qquad -1<y<1.$$

Write equation (10) as

$$u(x,y) = \int_{-\infty}^{\infty} e^{ixr} \frac{\sinh(1-y)r}{\sinh 2r} g_1(r)\,dr + \int_{-\infty}^{\infty} e^{ixr} \frac{\sinh(1+y)r}{\sinh 2r} g_1(r)\,dr.$$

where

$$g_1(r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itr}\,d\alpha(t).$$

Now apply equation (7) with $\alpha=\beta$:

$$\int_{-\infty}^{\infty} u(x,y)\,dx = \int_{-\infty}^{\infty} d\alpha(t) = 2\pi g_1(0) = \pi\phi(0).$$

In the example of §1, $\phi(0)=1$, so that equation (1) is established.

4. The Fourier sine-transform. A companion result to Theorem 2 is the following.

THEOREM 3. *The conditions*
A. $u(x,y) \in H, \in L\,(-\infty < x < \infty),\ -1 < y < 1,$
B. $u(x,y) \geqq 0,\ 0 < y < 1,$
C. $u(-x,y) = -u(x,-y) = u(x,y)$

are necessary and sufficient that

$$(11)\qquad u(x,y) = \operatorname{Im}\int_0^{\infty} \sin(x+iy)t\,\phi(t)\,dt, \qquad -1 < y < 1,$$

where $\phi(t)$ sinh t is real positive definite (ϕ being odd).

We first prove the necessity. Assume (11) with

$$\phi(t)\sinh t = g(t)\int_{-\infty}^{\infty} e^{-itr}\,d\alpha(r),$$

where $\alpha(r)$ is nondecreasing and bounded. Then

$$(12)\qquad u(x,y) = \frac{1}{2}\int_{-\infty}^{\infty} \cos xt\,\frac{\sinh yt}{\sinh t}\,g(t)\,dt$$

$$(13)\qquad = \frac{\pi}{2}\int_{-\infty}^{\infty} \frac{\sin \pi y}{\cosh(x-r)\pi + \cos \pi y}\,d\alpha(r).$$

From (13) we may now verify Conditions A and B; from (12), Condition C.

Conversely, from (9) with $c=1$, we have for $0<y<1$

$$(14)\quad u(x,y) = \int_{-\infty}^{\infty} e^{ixr}\frac{\sinh(1-y)r}{\sinh r}\,g(r)\,dr + \int_{-\infty}^{\infty} e^{ixr}\frac{\sinh yr}{\sinh r}\,h(r)\,dr,$$

where g and h are positive definite. Assuming $u(x, y)$ odd in y, we see that $u(x, 0)=0$ (A and C). We now show that the first of the integrals (14) is identically zero. From (13) it is clear that it defines a non-negative harmonic function $v(x, y)$ in $0<y<2$. The second of the integrals (14) is harmonic in $-1<y<1$ and vanishes for $y=0$. From equation (14), $v(x, 0)=0$. But $v(x, 1)=0$. Hence we may apply the uniqueness result, Corollary 2.2 of [1], to conclude that

$$v(x, y) = [Ae^{\pi x} + Be^{-\pi x}] \sin \pi y, \qquad A \geqq 0, B \geqq 0.$$

Since

$$\int_{-\infty}^{\infty} v(x, y)dx \leqq \int_{-\infty}^{\infty} u(x, y)\, dx < \infty, \qquad 0 < y < 1$$

it follows by Condition A that $A=B=0$, so that v vanishes identically.

Now writing $h=h_1+ih_2$ we have

$$\begin{aligned} u(x, y) &= \int_{-\infty}^{\infty} \cos xr \frac{\sinh yr}{\sinh r} h_1(r)\, dr \\ &= \operatorname{Im} \int_0^{\infty} \sin(x + iy)r\, \phi(r)\, dr, \\ \phi(r) = -\phi(-r) &= 2h_1(r)/\sinh r, \qquad 0 < r < \infty. \end{aligned}$$

This concludes the proof. As an example we may take

$$\begin{aligned} u(x, y) &= \frac{\pi}{2} \frac{\sin \pi y}{\cosh \pi x + \cos \pi y} = \frac{\pi}{2} \operatorname{Im} \tanh \pi(x + iy) \\ &= \operatorname{Im} \int_0^{\infty} \frac{\sin(x + iy)t}{\sinh t}\, dt. \end{aligned}$$

Here $\phi(t)=1/\sinh t$, and the function 1 is real and positive definite.

5. **Positive integrable harmonic functions in a half plane.** In [4] we showed that $u(x, y)$ is harmonic, $\geqq 0$ and integrable in x $(-\infty < x < \infty)$ for $0<y<\infty$ if and only if

$$u(x, y) = \int_{-\infty}^{\infty} e^{ixr - y|r|}\psi(r)\, dr, \tag{15}$$

where $\psi(r)$ is positive definite. Under these conditions $u(x, y)$ satisfies the conditions of Theorem 1 in the strip $0<y<c$ for every $c>0$. Thus $u(x, y)$ has the two representations (9) and (15). It is perhaps useful to record the relations between the functions g, h, and ψ. By use of the identity

$$e^{-y|r|} = e^{-c|r|}\frac{\sinh yr}{\sinh cr} + \frac{\sinh(c-y)r}{\sinh cr}$$

we see at once that

$$g(r) = \psi(r), \qquad h(r) = e^{-c|r|}\psi(r),$$

so that g is independent of c, and h is an exponential multiple of g. As one would expect the first integral (9) tends to the integral (15) as $c \to +\infty$, the second approaches zero.

References

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