INVERTIBLE SOLUTIONS TO THE OPERATOR EQUATION $TA-BT=C$

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1. **Introduction.** Let A, B and C be endomorphisms of the Banach space X. Consider bounded solutions to the operator equation

$$TA - BT = C. \tag{1}$$

If X is finite-dimensional, it is well known that for any C, a unique solution T of (1) exists provided that the eigenvalues of A are distinct from the eigenvalues of B [**1**]. An extension of this result has been given by Rosenblum [**2**]. For an arbitrary Banach space the operator equation (1) possesses a unique solution T provided that the spectrum of A is disjoint from the spectrum of B.

Certain results concerning the invertibility of T are available in the special case where X is finite-dimensional and (1) is replaced by

$$TA + A'T = C \tag{2}$$

where A is a stability matrix (all of its eigenvalues have negative real parts), and A' denotes the transpose of the matrix A. If C is a symmetric, negative definite matrix, it is well known [**3**] that the solution T of (2) is positive definite. Recently Kalman [**4**] has shown that if C is a symmetric dyad, $C=-cc'$, the solution T of (2) is positive definite provided the vectors c, Ac, $A^2c, \cdots, A^{n-1}c$, are linearly independent.

In this note, Kalman's result is generalized to apply to equation (1) when C is an operator with one-dimensional range. Necessary and sufficient conditions are given which guarantee that the unique solution T has nullspace equal to the null vector and range dense in X. Unlike the methods used in [**4**] which rely on finite-dimensionality through use of canonical forms, the methods used here apply in an arbitrary Banach space.

2. **Definitions and notation.** The dual space of X is denoted by X^*. The null vectors in X and X^* are denoted by θ and θ^* respectively. Corresponding to a subspace V of X is the subspace $V^\perp$ of X^* which consists of all elements $x^* \in X^*$ such that $x^*(v)=0$ for each $v \in V$.

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The adjoint of an endomorphism A is denoted by A^*. The range and nullspace of A are denoted by $\mathcal{R}(A)$ and $\mathfrak{N}(A)$ respectively. The resolvent set $\rho(A)$ of A is the set of complex numbers λ such that $(\lambda-A)^{-1}$ exists as an endomorphism of X. The spectrum of $A, \sigma(A)$, is the complement of the resolvent set in the complex plane.

3. **Invertible solutions.** The results of this section characterize the invertible solutions of equation (1). The characterization is in terms of the conditions of Theorem 1 together with the requirement that the range of T be closed.

Rosenblum [2] provides an explicit representation for the solution T of (1).

LEMMA. *If $\sigma(A)$ and $\sigma(B)$ are disjoint sets, then* (1) *has the unique solution*

$$T = \frac{1}{2\pi i}\int_{b(D)} (\lambda - B)^{-1}C(\lambda - A)^{-1}\, d\lambda \tag{3}$$

where D is a Cauchy domain such that $\sigma(A) \subset D$ and $\sigma(B) \cap \overline{D} = \emptyset$.

PROOF. See Rosenblum [2, Theorem 3.1].

THEOREM 1. *Suppose that $\sigma(A)$ and $\sigma(B)$ are disjoint sets and that C has a one-dimensional range. Then the unique solution T of* (1) *has nullspace equal to the null vector and range dense in X, if and only if*

(i) *the largest A-invariant subspace of X contained in the nullspace of C is the space consisting of θ alone, and*

(ii) *the smallest B-invariant closed subspace of X containing the range of C is X.*

PROOF. The representation (3) for T shows immediately that any A-invariant subspace of X which is contained in the nullspace of C is also contained in the nullspace of T. Hence condition (i) is certainly a *necessary* condition.

Condition (ii) is equivalent to:

(ii)′ The largest B^*-invariant subspace of X^* contained in the nullspace of C^* is the space consisting of θ^* alone.

The equivalence of (ii) and (ii)′ follows from the relation $\mathcal{R}(C)^\perp = \mathfrak{N}(C^*)$ and the fact that if W is a B-invariant subspace in X, $W^\perp$ is a B^*-invariant subspace in X^*.

If T satisfies (1) on X then T^* satisfies

$$A^*T^* - T^*B^* = C^* \tag{4}$$

on X^*. T has nullspace equal to the null vector and the closure of its range equal to X if and only if the same is true of T^*. Hence, the argument above which shows that condition (i) is necessary may be applied to equation (4) to show that condition (ii)′ is necessary. Hence, by the equivalence of (ii) and (ii)′, (ii) is a *necessary* condition.

To conclude the proof, it must be shown that if either the nullspace of T is not $\{\theta\}$ or the closure of the range of T is not X either condition (i) or (ii) is unsatisfied. Suppose the nullspace of T is not $\{\theta\}$. Let $x\neq\theta$ belong to the nullspace of T. Then $TAx=Cx$. Either $Cx\neq\theta$ or $Ax\in\mathfrak{N}(T)$. The second alternative together with equation (1) implies that $TA^2x=CAx$. This last equation implies that either $CAx\neq\theta$ or $A^2x\in\mathfrak{N}(T)$. Continuation in this manner, by induction, leads to the conclusion: either there exists an $x_0\in\mathfrak{N}(T)$ such that $Cx_0\neq\theta$ or, for all positive integers n, $A^nx\in\mathfrak{N}(C)$.

If $A^nx\in\mathfrak{N}(C)$ for all n, the A-invariant subspace $\{x, Ax, \cdots, A^nx \cdots\}$ violates condition (i).

If there exists an $x_0\in\mathfrak{N}(T)$ with $Cx_0\neq\theta$, the Fredholm alternative and (1) imply that $\mathfrak{N}(A^*T^*)$ is perpendicular to Cx_0. Since $\mathfrak{R}(C)$ is one-dimensional, $\mathfrak{N}(A^*T^*)$ is perpendicular to $\mathfrak{R}(C)$. Hence, if $x^*\in\mathfrak{N}(A^*T^*)$, then $x^*\in\mathfrak{N}(C^*)$. Equation (4) on X^* then implies that $T^*B^*x^*=\theta^*$ which in turn implies that $(A^*T^*)B^*x^*=\theta^*$. Hence, the nullspace of A^*T^* is B^*-invariant, and the nullspace of C^* contains the nullspace of A^*T^*. This violates condition (ii)′.

Suppose that the closure of the range of T is not X. This implies that the nullspace of T^* is not $\{\theta^*\}$. It is clear that (4) on X^* satisfies the hypotheses of the theorem so that arguments similar to those above applied to (4) show that if the nullspace of T^* is not $\{\theta^*\}$, condition (i) or (ii) must be violated. This concludes the proof of the theorem.

If X is n-dimensional and A, B, and C are $n\times n$ matrices, then C may be represented in the form $C=ba'$ where b is a column vector and a' is a row vector. In this case, Theorem 1 becomes

THEOREM 2. *Suppose the eigenvalues of A are distinct from the eigenvalues of B. Then the unique solution T of the equation $TA-BT=ba'$ is invertible if and only if*

(i)″ *the (row vectors) $a', a'A, \cdots, a'A^{n-1}$ are linearly independent, and*

(ii)″ *the (column vectors) $b, Bb, \cdots, B^{n-1}b$ are linearly independent.*

PROOF. For n-dimensional X, conditions (i) and (ii) are equivalent to conditions (i)″ and (ii)″, and clearly the range of T is closed.

References

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