A CONDITION FOR FINITE DIMENSIONAL CONVEXITY

D. G. BOURGIN[1]

This note is devoted to a demonstration of

THEOREM. *Let K be a compact contractible subset of R^N. Suppose every support set is acyclic. Then K is convex. The converse is obvious.*

Here R^N is N-dimensional Euclidean space. We use Cech homology with integer coefficient group [1]. *Plane* will be used for $N-1$ dimensional hyperplane. A *support plane*, P, is a plane meeting K where K is in just one of the two closed half spaces determined by P. $K \cap P$ is the *support set* for P. A weaker theorem, requiring the added hypothesis of local contractibility for K was proved by Kuhn [2] who pointed out this hypothesis was essential for any proof based on his methods. Liberman [3] has established a slightly weaker theorem requiring contractibility for the support sets as the culmination of a long chain of subsidiary results.

PROOF. We need R^n, the linear extension of K, so $n \leqq N$. To avoid triviality *we assume* $n > 1$. Let S^{n-1} and Y refer to the $n-1$ topological sphere and to the metric unit $n-1$ sphere respectively. Each point y in Y determines a unique supporting hyperplane P_y whose normals pointing away from K are parallel translates in R^n of the vector $\mathbf{oy}$. We write X for the set of boundary points of K and X_y for the support set $P_y \cap X$.

Let

$$\Gamma = \{(y, x) \mid x \in X_y\} \subset Y \times X.$$

For each x we introduce a subset of Y,

$$Y_x = \{y \mid (y, x) \in \Gamma\}.$$

Let p and q project Γ into Y and into X respectively. Thus

$$p: y \times X_y = y.$$
$$q: Y_x \times x = x.$$

The image of Γ under p covers Y while that under q is $X' \subset X$. Obviously X and X' are closed in R^n. Let $\hat{K}$ be the convex hull of K and denote its boundary by $\hat{K}^\bullet$. It is important to note that:

(a) $$X' \subset \hat{K}^\bullet.$$

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Observe that Y_x is not affected if X' is replaced by $\hat{K}$. This is to say the join OY_x of O and Y_x is the translation to origin O of the normal cone to $\hat{K}$ at x. The set $Y_0 \subset Y$ is said to be sphere convex if every pair of distinct points in Y_0 are endpoints of a unique great circle arc contained in Y_0. It is well known that the normal cone at a boundary point of a convex set is closed and convex. Hence OY_x is closed convex. The use of R^n (in place of R^N) assures the absence of antipodal points in Y_x. Accordingly Y_x is closed and sphere convex. It is therefore obviously contractible over Y and so acyclic. (If the $N-1$ unit sphere is used for Y, where $N>n$, then Y_x need not be acyclic.)

Both p and q are continuous and $p^{-1}y$ is homeomorphic to the acyclic support set X_y while $q^{-1}x$, for $x \in X'$, is homeomorphic to the acyclic set Y_x. The Vietoris-Begle Theorem [1, p. 503] guarantees therefore that

$$H_*(X') \approx H_*(\Gamma) \approx H_*(Y). \tag{b}$$

We remark also that $\hat{K}^{\bullet}$ is of course an S^{n-1}. Thus

$$H_*(Y) \approx H_*(\hat{K}^{\bullet}) \approx H(S^{n-1}). \tag{c}$$

From (b) and (c) results

$$H_*(X') \approx H_*(S^{n-1}).$$

However since a (closed) proper subset of a topological $n-1$ sphere has a trivial $n-1$ homology group, it follows from (a) that

$$X' = \hat{K}^{\bullet}.$$

A Euclidean $n-1$ sphere, $n>1$, is not contractible over a proper subset of the disk it bounds. Accordingly, if X', the homeomorph of such a sphere is a proper subset of X, then K is a proper subset of $\hat{K}$ and so X' is not contractible over K and a fortiori neither is K in contradiction with our hypothesis.

One may conjecture that a similar theorem is valid in a Banach space.

Bibliography

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University of Illinois