ON DENSE SUBSPACES OF MOORE SPACES

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By a development for a topological space is meant a sequence of collections of basis elements (called regions) satisfying conditions 1, 2, and 3 of Axiom 1 of [4]; by a complete development is meant a development which satisfies condition 4 of this axiom. A (complete) Moore space is a topological space which has a (complete) development. A Moore space is completable if and only if some complete Moore space contains it as a subspace. Clearly if S is completable, then S is dense in some complete Moore space S', which will be called a completion of S. A development G for S satisfies Axiom C at the point p of S if and only if, for every region R containing p there is an integer n such that every element of G_n which intersects an element of G_n containing p is a subset of R. Younglove proved [7] that every complete development for a complete Moore space S satisfies Axiom C at each point of a dense subset M of S, so that M, regarded as space, satisfies Axiom C and is thus metrizable [5]. In this note, it is shown that every completable Moore space contains a dense metrizable subspace. It is not true, however, that every development of a completable Moore space (even a metrizable space) satisfies Axiom C at some point. It is proved that in order for some development for S to satisfy Axiom C at each point of a dense subspace of S, it is necessary and sufficient that S contain a dense subspace which is strongly screenable in S. Throughout this note certain terminology and theorems from [4] are used without explicit mention.

LEMMA 1. *If S is a topological space, M is a dense subset of S, and U and V are mutually exclusive domains with respect to M, then there exist mutually exclusive domains D_U and D_V in S, containing U and V, respectively.*

THEOREM 1. *If S is a completable Moore space, then every subspace of S contains a dense metrizable subspace.*

PROOF. It suffices to prove that every completable Moore space contains a dense metrizable subspace. Suppose S is a Moore space, and T is a completion of S. Then S is dense in T. Let G denote a complete development for T. Let G_1' denote a maximal collection of mutually exclusive regions of G_1. Now every region in T intersects S.

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Denote by K_1 a subset of S which intersects each element of G_1' at only one point; $K_1 \subset G_1'^*$. Denote by G_2' a maximal collection of mutually exclusive regions of G_2 whose closures are subsets of elements of G_1', which is such that K_1 is a subset of $G_1'^*$. Let K_2 denote a subset of $S \cdot G_2'^*$ which contains K_1 and intersects each element of G_2' at only one point. Continue this process, obtaining sequences G' and K, such that K_n is a subset of $S \cdot G_n'^*$ containing only one point of each element of G_n', $K_n \subset K_{n+1}$, and G_n' is dense in T. Let $M = K_1 + K_2 + \cdots$. Then by a part of the argument for Theorem 164 of [4], M is dense in T. Clearly, then, M is dense in S. For each positive integer n, let H_n denote the collection to which the point set h belongs if and only if, for some integer $m \geqq n$ and some element g of G_m', h is $g \cdot M$. Then $H = (H_1, H_2, H_3, \cdots)$ is an Axiom C development for M, so that M is metrizable [5].

THEOREM 2 (RUDIN). *There exists a Moore space which contains no dense metrizable subspace.*

PROOF. Mary Ellen Estill Rudin [6] showed the existence of a Moore space which is not separable but in which every collection of mutually exclusive domains is countable. Consider such a space S. Suppose it contains a dense metrizable subspace M. Let G be a collection of mutually exclusive domains in M. There exists, by Lemma 1, a collection G' of mutually exclusive domains in S covering G, each element of which contains only one element of G. By hypothesis G' is countable. Therefore G is also. Since M is metrizable, and every collection of mutually exclusive domains in M is countable, then M is separable. Denote by K a countable dense subset of M. Then K is a countable dense subset of S. This involves a contradiction.

REMARK. In [2], D. R. Traylor and the author constructed, starting with a Moore space S^0, a Moore space S^W such that (1) if α is an infinite cardinal, S^0 is α-separable if and only if S^W is, (2) S^0 is normal if and only if S^W is, (3) every open set in S^W contains a copy of S^0. Thus if S^0 is not metrizable (completable), S^W is not locally metrizable (completable) at any point.

THEOREM 3 (HEATH). *There exists a Moore space S with a dense, topologically complete metrizable subspace M such that no development for S satisfies Axiom* C *at each point of M.*

PROOF. R. W. Heath [3] constructed an example of a nonmetrizable Moore space S which is the sum of two topologically complete metrizable subspaces S_1 and S_2 each dense in S. Suppose there are developments G and H for S which satisfy Axiom C at each point of

S_1 and S_2, respectively. There is a development I for S which is a common refinement of G and H. Then I satisfies Axiom C at each point of S_1 and S_2.

THEOREM 4 (YOUNGLOVE). *There exists a metrizable space with a development not satisfying Axiom* C *anywhere.*

PROOF. Younglove proved [7] that if the Moore space S is not compact, but is complete, M is a dense inner limiting set in S, and some development for S satisfies Axiom C at each point of M, then some development for S satisfies Axiom C at each point of M and at no other point. Consider the line E^1; there exists a development which satisfies Axiom C everywhere; the irrationals, I, form a dense inner limiting set in E^1; so there is a development G for E^1 which satisfies Axiom C at each point of I and at no other point of E^1. Let R denote the rationals, and for each n, let G_n' denote the collection to which g' belongs if and only if g' is $g \cdot R$ for some g in G_n. Then G' is clearly a development for the subspace R. Now if x is in R, there exists a domain D containing x such that, for each positive integer n, there exist regions g_n and h_n of G_n such that x is in g_n, h_n intersects g_n and contains a point not in $\overline{D}$. In the subspace R, let $D' = D \cdot R$, $g_n' = g_n \cdot R$, and $h_n' = h_n \cdot R$. Since R is dense in E^1, every domain intersects R. Now g_n' and h_n' belong to G_n', x is in g_n', h_n' intersects g_n', and h_n' contains a point not in D'. Thus G' satisfies Axiom C nowhere.

DEFINITION. The subset M of the topological space S is said to be strongly screenable in S if and only if, for each collection of domains G in S covering M, there exist discrete collections $H_1, H_2, \cdots$ of mutually exclusive domains in S such that for each i, H_i is a refinement of G and $\sum H_i^* \supset M$.

THEOREM 5. *In a Moore space S, the following are equivalent*:

(i) *There exists a development for S which satisfies Axiom* C *at each point of a dense subset.*

(ii) *There exists a dense subset of S which is strongly screenable in S.*

PROOF. Suppose (i) is true, and G is such a development, and M is such a dense subspace of S. There exists a maximal subcollection G_1'' of G_1 such that no region of G_1 intersects two regions of G_1''. Note that if R is in G_1 some region of G_1 intersects both R and $G_1''^*$. Let K_1 denote a subset of M containing only one point of each element of the discrete collection G_1'. Let G_2' denote the set of all regions g of G_2 such that $\bar{g}$ is a subset of $S - G_1'^*$ or of some element of G_1''. There exists a maximal subcollection G_2'' of G_2' such that no region of G_2 intersects two regions of G_2'', and such that K_1 is a subset of $G_2''^*$.

Continue this process, obtaining sequences G_1'', G_2'', G_3'', $\cdots$, and $K_1, K_2, K_3, \cdots$ such that each G_n'' is a discrete subcollection of G_n such that no region of G_n intersects two regions of G_n'' but if R is in G_n some region of G_n intersects both R and $G_n''^*$, and such that K_n is a subset of K_{n+1} and of $G_n''^*$, and of M, and K_n contains only one point of each element of G_n''. Let $K=K_1+K_2+\cdots$. Suppose K is not dense in S. There is a region R which does not intersect $\overline{K}$ but which does intersect M at some point x. There exists a positive integer n such that if g is in G_n and contains x, h is in G_n and intersects g, and k is in G_n and intersects h, then k is a subset of R. But some region of G_n contains x and intersects a region of G_n intersecting $G_n''^*$. Let k be a region of G_n'' which is a subset of R. Then $k\cdot K_n$ intersects R. Thus K is dense in S.

Suppose I is a collection of domains covering K. Let H_n denote the set of all regions of G_n'' that are subsets of elements of I. Then H_n is, for each n, a discrete refinement of I. Moreover, $H_1^*+H_2^*+\cdots$ contains K, for suppose x is in K_n. Thus for each $m\geqq n$, x is in some element of G_m''. Some region R of G contains x. There is an integer $m\geqq n$ such that every region of G_m that contains x is a subset of R. Then x is in H_n^*. Thus K is strongly screenable in S.

Now suppose (ii) is true and M is a dense subset of S that is strongly screenable in S. Let G denote a development for S. Let K_1 denote a maximal subset of M such that no region of G_1 contains two points of K_1. Let $H_1, H_2, \cdots$ be a sequence of discrete refinements of G_1 covering M. Let $K_{1i}=H_i^*\cdot K_1$. Then K_{1i} is a closed and isolated point set (i.e., no point of it is a limit point of it) such that some discrete collection of regions covers it, and each element of that discrete collection contains only one element of it. Similarly, define $K_2, K_3, \cdots$. Thus there is a dense subset K of S which is the sum of countably many point sets K_{ij} such that each is covered by a discrete collection of regions intersecting it at only one point. It suffices to prove that each such point set has Younglove's property Q [7] for if so there is a development G^{ij} which satisfies Axiom C at each point of K_{ij}, and one development refining all of these, so that there is a development satisfying Axiom C at each point of K. So now suppose that L is a closed and isolated point set, that there exists a discrete collection H_1 of regions covering L, and that G is a collection of domains covering S. Let H_2 denote a discrete refinement of H_1 and of G covering L. Let H_3 denote a discrete collection of regions covering L the closure of each of which is a subset of some element of H_2. For each point x in $S-H_2^*$ there is a region g_x which contains x, is a subset of some element of G and does not intersect H_3^*. Let H_4 be the collection to which g belongs if and only if, for some x in $S-H_2^*$, g is g_x. Then H_2+H_4 covers S and is locally finite at each point of L. Thus L has Younglove's property Q.

The author does not know whether the existence of a dense metrizable subspace of a Moore space implies conditions (i) and (ii) of Theorem 5, although this is the case if S is normal. To see this, consider a development G for S, and let M denote a dense metrizable subspace. There exists a sequence K such that, for each n, $K_n \subset K_{n+1}$, no region of G_n contains two points of K_n, if x is in M some region of G_n contains x and intersects K_n. Let $L=K_1+K_2+\cdots$; then L is metrizable and dense in S. Also, each K_i is a closed and isolated subset of L and thus has an open covering of mutually exclusive regions, each containing only one point of K_i. Then by Lemma 1 there is, in S, such an open covering of K_i. Using normality, one can obtain a discrete open covering of each K_i and, as in Theorem 5, K_i has Younglove's property Q; so that some development satisfies Axiom C at each point of K_i. It follows that some development satisfies Axiom C at each point of L.

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