

A UNIVERSAL SNAKE-LIKE CONTINUUM¹

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1. Introduction. By a *snake-like continuum* we mean a compact metric chainable continuum. In [2], Mioduszowski discusses snake-like continua and calls a pseudo-arc the universal snake-like continuum since, as proved in [2], every snake-like continuum is a continuous image of a pseudo-arc. This usage of "universal" is at variance with the homeomorphic imbedding usage of universal and thus the question arises: Is there a universal snake-like continuum X in the sense that every snake-like continuum can be imbedded in X ? In this paper we construct such a universal snake-like continuum. This construction is based on the well-known theorem that X is snake-like if and only if X is the limit of an inverse sequence of arcs.

2. The construction. Let I denote the closed unit interval with the usual topology and let $C(I)$ be the space of all continuous functions from I into I with the topology of uniform convergence. $C(I)$ is a separable metric space and thus there exists a countable dense subset $\{u_1, u_2, \dots\}$ of $C(I)$.

If n is a positive integer, let I_n denote I and for each n greater than one we construct a continuous function $g_n: I_n \rightarrow I_{n-1}$. The limit U of the resulting inverse sequence (I_i, g_i) will be a universal snake-like continuum.

Let $u_0: I \rightarrow I$ be the identity function and if i is an integer, define $E(i)$ such that $E(i) = i/2$ if i is even, and $E(i) = (i-1)/2$ if i is odd. If $n \geq 2$ and i is an integer such that $0 \leq i < 2^{n-1}$, let

$$(*) \quad g_n(x) = 2^{2-n}E(i) + 2^{1-n}u_{(n-1)[i \bmod 2]}(2^n x - 2i)$$

for all x in $[2i2^{-n}, (2i+1)2^{-n}]$. If $0 < i < 2^{n-1}$, let

$$(**) \quad g_n(x) = \frac{g_n(2i2^{-n}) - g_n((2i-1)2^{-n})}{2i2^{-n} - (2i-1)2^{-n}}(x - 2i2^{-n}) + g_n(2i2^{-n})$$

for x in $[(2i-1)2^{-n}, 2i2^{-n}]$, and suppose $g_n(x) = g_n((2^n-1)2^{-n})$ for x in $[(2^n-1)2^{-n}, 1]$.

Thus, we have divided I_n into 2^n equal subintervals and by (*) we have defined a continuous function g_n on the odd subintervals. The

Presented to the Society, April 25, 1964; received by the editors August 7, 1964.

¹ This paper is part of the author's doctoral dissertation at the University of Iowa, prepared under the supervision of Professor S. Armentrout while the author was supported on an NSF Cooperative Fellowship.

endpoints of the even subintervals have been defined and thus, by (**), we make linear extensions of g_n to the even subintervals. Hence, g_n is a continuous function from I_n into I_{n-1} .

Hence, for all i greater than one we have defined a continuous function $g_i: I_i \rightarrow I_{i-1}$. Let (I_i, g_i) be the corresponding inverse sequence of arcs and hence the limit space U is a snake-like continuum.

THEOREM. *U contains a homeomorphic copy of every snake-like continuum.*

PROOF. Let X be a snake-like continuum and let (I_i, f_i) be an inverse sequence of arcs whose limit is X . Theorem 3 of Morton Brown's paper [1] essentially says that if we approximate each of the functions f_i close enough, then we still obtain X as the limit. Since $\{u_1, u_2, \dots\}$ is a dense subset of $C(I)$, the u_i can be used to approximate the f_i and thus, with the aid of [1], it follows that there exists an increasing sequence (n_2, n_3, \dots) of positive integers such that (1) u_{n_i} approximates f_i and (2) X is homeomorphic to the limit Y of the inverse sequence (I_i, u_{n_i}) . It is clear that there is a sequence (n_2, n_3, \dots) such that if $i \geq 2$ then $n_i + 1 < n_{i+1}$.

We now show that there exists a subspace of U homeomorphic to Y and thus to X . Let J_1 denote the subinterval $[0, 1/2]$ of I_1 , let $n_1 = 0$, and $\sum_{j=2}^1 2^{-n_j} = 0$. If $k = 2, 3, \dots$, let J_k be the subinterval

$$\left[\sum_{j=2}^k 2^{-n_j}, \sum_{j=2}^k 2^{-n_j} + 2^{-n_{k+1}} \right] \text{ of } I_{n_{k+1}}.$$

Also define $w_k: J_k \rightarrow I_{n_{k-1}+1}$ such that

$$w_k = g_{n_{k-1}+2} \circ \dots \circ g_{n_{k+1}} | J_k.$$

For instance, if $n_2 = 1$ and $n_3 = 3$, then J_2 and J_3 are the respective subintervals $[1/2, 3/4]$ and $[10/16, 11/16]$ of I_2 and I_4 . Thus, $w_2 = g_2 | J_2$ is the compressed u_1 and $w_3 = g_3 g_4 | J_3$ is the compressed u_3 followed by the compressed identity.

LEMMA. $w_k(x) = \sum_{j=2}^{k-1} 2^{-n_j} + 2^{-n_{k-1}-1} u_{n_k} (2^{n_{k+1}} x - 2^{n_{k+1}} \sum_{j=2}^k 2^{-n_j})$.

The proof of the lemma follows the proof of the theorem.

Thus we see that w_k is essentially u_{n_k} and furthermore, that $w_k [J_k] \subset J_{k-1}$. Hence, we have defined the inverse sequence (J_i, w_i) and will prove that its limit, W , is homeomorphic to Y . Clearly W is a subset of U . If $k = 1, 2, \dots$, define the homeomorphism $q_k: J_k \rightarrow I_k$ as follows:

$$q_k(x) = 2^{n_{k+1}} x - 2^{n_{k+1}} \sum_{j=2}^k 2^{-n_j}.$$

By straightforward calculations, we see that the diagram

$$\begin{array}{ccccccc}
 J_1 & \xleftarrow{w_2} & J_2 & \xleftarrow{\dots} & J_{k-1} & \xleftarrow{w_k} & J_k & \xleftarrow{\dots} \\
 \downarrow q_1 & & \downarrow q_2 & & \downarrow q_{k-1} & & \downarrow q_k & \\
 I_1 & \xleftarrow{u_{n_2}} & I_2 & \xleftarrow{\dots} & I_{k-1} & \xleftarrow{u_{n_k}} & I_k & \xleftarrow{\dots}
 \end{array}$$

is commutative. That is, if $k = 2, 3, \dots$, then $q_{k-1}w_k = u_{n_k}q_k$. Therefore we have a mapping $(q_i): (J_i, w_i) \rightarrow (I_i, u_{n_i})$ which induces a homeomorphism $q: W \rightarrow Y$. Hence, we have a subspace W of U that is homeomorphic with the snake-like continuum X and thus U is a snake-like continuum that contains a homeomorphic copy of every snake-like continuum.

PROOF OF THE LEMMA. If $i = 2^{n_k} \sum_{j=2}^k 2^{-n_j} = (\sum_{j=2}^{k-1} 2^{n_k-n_j}) + 1$, then i is an odd integer since $n_k - n_j > 0$ where $j = 1, \dots, k-1$. Thus,

$$J_k = [2i2^{-n_{k-1}}, (2i + 1)2^{-n_{k-1}}]$$

and

$$E(i) = \frac{1}{2} \sum_{j=2}^{k-1} 2^{n_k-n_j} = 2^{n_{k-1}} \sum_{j=2}^{k-1} 2^{-n_j}$$

and hence, by the definition of $g_{n_{k+1}}$,

$$g_{n_{k+1}}|J_k = \sum_{j=2}^{k-1} 2^{-n_j} + 2^{-n_k}u_{n_k} \left(2^{n_{k+1}}x - 2^{n_{k+1}} \sum_{j=2}^k 2^{-n_j} \right).$$

With similar calculations including an inductive step, we show that if $0 \leq p \leq n_k - n_{k-1} - 2$, then

$$\begin{aligned}
 g_{n_{k-p}} \circ \dots \circ g_{n_{k+1}}|J_k &= \sum_{j=2}^{k-1} 2^{-n_j} + 2^{-n_{k+p+1}}u_{n_k} \left(2^{n_{k+1}}x - 2^{n_{k+1}} \sum_{j=2}^k 2^{-n_j} \right).
 \end{aligned}$$

In particular, it is true for $p = n_k - n_{k-1} - 2$, and thus

$$w_k(x) = \sum_{j=2}^{k-1} 2^{-n_j} + 2^{-n_{k-1}-1}u_{n_k} \left(2^{n_{k+1}}x - 2^{n_{k+1}} \sum_{j=2}^k 2^{-n_j} \right).$$

COROLLARY. If X is any snake-like continuum, then U contains c mutually disjoint homeomorphic copies of X .

PROOF. Just as there are c distinct subsequences of the positive integers, there are c ways of choosing the sequence (n_2, n_3, \dots) in

the proof of Theorem 1. The result follows since distinct subsequences yield disjoint subspaces of W .

Added in proof. With a slight modification U can also be universal in the Mioduszowski continuous image sense. The point $(0, 0, \dots)$ of U determined by the inverse limit representation is an endpoint and thus we can add a pseudo-arc P such that $P \cup U$ is snake-like. If X is any snake-like continuum, let f map P onto X by [2] and extend f to $P \cup U$ by collapsing U to the appropriate point.

BIBLIOGRAPHY

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A NOTE ON COUNTABLY PARACOMPACT SPACES AND METRIZATION

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As defined by Dowker [1], a topological space (X, \mathfrak{J}) is called countably paracompact if every countable open cover of X has an open locally finite refinement. It is known that every metrizable topological space is paracompact and hence countably paracompact.

It is proposed to show that in the usual metrization theorem for topological spaces [3, p. 127], the T_2 condition may be replaced by the combination T_2 and countably paracompactness. We will need the following lemma.

LEMMA. *Every countably paracompact, first countable T_2 space is T_3 .*

PROOF. Designate the topological space by (X, \mathfrak{J}) as in Kelley [3]. Let F be a proper closed set of X . For $x \notin F$, and $y \in F$ there exists disjoint open set $U_x^{(y)}$ and U_y containing x and y respectively. There exists a countable base at x , $\{V_k\}$, $k=1, 2, \dots, n, \dots$. Let $W_k = \bigcup \{U_y: U_x^{(y)} \supset V_k\}$. $W_k \cap V_k = \emptyset$ and $\{W_k\}$ forms a countable cover of F . Since F is closed, $\{W_k\}$ may be replaced by an open locally finite refinement, $\{T_a: a \in A\}$ where A is an index set.

Received by the editors December 28, 1964.