CONTRACTIBLE COMPLEXES IN $S^n$

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1. Introduction. By a pseudo $n$-cell is meant a contractible compact combinatorial $n$-manifold with boundary. Poenaru [9] and Mazur [7] gave the first examples of pseudo 4-cells which are not topological 4-cells, but whose products with the unit interval are topologically 5-cells. Newman [8] defines a 2-complex $P$ such that $\pi_1(P) \neq 1$, while $H_1(\mathbb{Z}, Z) = 0 = H_2(\mathbb{Z}, Z)$. Curtis [4] making use of this 2-complex has shown that, for each $n \geq 4$, there exists a pseudo $n$-cell which is not a topological $n$-cell because its boundary fails to be simply connected. Curtis [4] also shows that the cartesian product of a pseudo $n$-cell and an interval is the topological $(n+1)$-cell, $I^{n+1}$ if $n \geq 5$.

Curtis [5] making use of Mazur's peculiar embedding of the dunce hat in $S^4$ [7], [13] gives an example of a contractible 2-complex $K$ embedded as a subcomplex of a combinatorial triangulation of $S^4$ such that $\pi_1(S^4 - K) \neq 1$. The purpose of this paper is to show that for $n \geq 4$ there exists a contractible $(n-2)$-complex $K^{n-2}$ combinatorially embedded in $S^n$ such that $\pi_1(S^n - K^{n-2}) \neq 1$. The regular neighborhood $N^n = N(K^{n-2})$ of $K^{n-2}$ in $S^n$ is also a pseudo $n$-cell which fails to be a topological $n$-cell and its product with the unit interval $I$ is shown to be a combinatorial $(n+1)$-cell, rather than just mere topology. In addition, each $N^n$ ($n \geq 5$) gives examples of combinatorial $n$-manifolds with boundary which are not topologically $I^n$ but can be expressed as the union of two combinatorial $n$-balls whose intersection is also a combinatorial $n$-ball.

2. Definitions. We will use the terminology of [12], [13]. All manifolds and all mappings or homeomorphisms will be considered in the combinatorial sense. We will use $\approx$ to denote combinatorial equivalence. If the complex $K$ collapses to the complex $L$, this will be denoted $K \searrow L$.

Let $f: X \to Y$ be continuous. The identification space $Y_f$ derived from $(X \times [0, 1]) \cup Y$ by identifying each point $(x, 1)$ with the point

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$f(x)$ in $Y$ and using the identification topology is called the mapping cylinder of $f$.

3. Preliminaries. The following two lemmas are well known and elementary, hence no proof will be included.

**Lemma 1.** If $C$ is a $k$-complex embedded as a finite subcomplex of a combinatorial $n$-sphere $S^n$ and $M$ is a regular neighborhood of $C$ in $S$, with $C \subset \text{int } M$, then there is a combinatorial map $\phi: \partial M \to C$ such that $M$ is combinatorially equivalent to $I \times \partial M \cup \phi C$, the mapping cylinder of $\phi$.

**Lemma 2.** Suppose $C$ is a $k$-complex embedded as a finite subcomplex of a combinatorial $n$-sphere $S^n$ and $N$ is any regular neighborhood of $C$ in $S^n$, such that $C \subset \text{int } N$; then $\pi_1(N - C) = \pi_1(\partial N)$.

The topological dunce hat $D$ is obtained from a triangle $abc$ say, by identifying all three sides $ab = bc = ac$.

**Theorem 1.** There exist two combinatorially inequivalent embeddings $D_1, D_2$ of the dunce hat $D$ in $S^4$, such that the regular neighborhood $N_1$ of $D_1$ is combinatorially $I^4$, while $N_2$ the regular neighborhood of $D_2$ is not topologically $I^4$. Moreover, $\pi_1(\partial D_1) \neq 1$, $\pi_1(N_2 - D_2) \neq 1$, but $\pi_1(S^4 - D_2) = 1$.

**Proof of Theorem 1.** Let $D_1$ be any combinatorial embedding of $D$ in $S^4 \subset S^4$. Then $N_1 \cap N_1$, the regular neighborhood of $D$ in $S^4$, and since $N_1 \approx I^4$, $N_2 \approx I^4$.

For $D_2$ we will use Mazur’s embedding of $D$ in $S^4$ (as in Theorem 5 [13]). Since $N_2 \approx \partial W^4$ (also Theorem 5 [13]) and $\pi_1(\partial W^4) \neq 1$ (see [7]) we have that $N_2 \neq I^4$. The fact that $\pi_1(N_2 - D_2) \neq 1$ follows from Lemma 2. We see that $\pi_1(S^4 - D_2) = 1$ by considering Mazur’s embedding of $D$ in $S^4$. That is $D \subset \partial W^4 \subset \partial W^4 = S^4$. Since $S^4 - D_2 \approx \partial W^4 \cup W^4 - D_2 \approx \partial W^4 \cup (\partial W \times [0, 1])$ (using Lemma 1), we see that $S^4 - D_2$ is of the same homotopy type as $W^4$ and $\pi_1(S^4 - D_2) = 1$.

To see that these two embeddings are combinatorially inequivalent, suppose there exists a p.w.l. homeomorphism taking $S^4$ onto $S^4$ carrying $D_1$ onto $D_2$. Let $a_1, a_2$ be the points of $D_1, D_2$ respectively, which correspond to the point $a(=b=c)$ in $D$. Then by subdividing the triangulation of $S^4$ so that $h$ is simplicial, we get that $h$ carries $\text{st}(a_1, S^4)$ onto $\text{st}(a_2, S^4)$, each combinatorial 4-balls. Also $h$ carries $\text{lk}(a_1, D_1) \subset \text{lk}(a_1, S^4) \approx S^4$ onto $\text{lk}(a_2, D_2) \subset \text{lk}(a_2, S^4) \approx S^4$. This leads to a contradiction, since there exists no homeomorphism of $S^4$ onto $S^4$ carrying $\text{lk}(a_1, D_1)$ as in $\text{lk}(a_1, S^4)$ onto $\text{lk}(a_2, D_2)$ as in $\text{lk}(a_2, S^4)$.

See Figures 5 and 8 of [13].
Theorem 2. There exists a contractible 2-complex $K$ and two inequivalent embeddings $K_1, K_2$ of $K$ in $S^4$ so that the regular neighborhood $N_1$ of $K_1$ is a combinatorial 4-ball, while $\pi_1(S^4 - K_2) \neq 1$.

Remark. Since $N_1 \approx I^4$, $K_1$ is cellular in $S^4$ and hence $S^4 - K_1 = E^4$ and $\pi_1(S^4 - K_1) = 1$. Also it will follow from a later result, which does not use the particular construction of the embedding of $K_2$ in $S^4$, that if $N_2$ is the regular neighborhood of $K_2$ in $S^4$ then $\pi_1(Bd N_2) \neq 1$ and hence $N_2 \approx I^4$.

Proof of Theorem 2. $K$ will be the union of two disjoint copies of the dunce hat $D$ joined together by a polyhedral segment intersecting each in $a (= b = c)$. $K_1$ will be the embedding of $K$ in $S^4 \subset S^4$ and $N_1 \approx I^4$ as in Theorem 1.

To get $K_2$, we will use Curtis's modification [5]. Let us again consider $S^4$ as $2W^4$ (Mazur's pseudo 4-cell). We have a $D'$ and $D''$ (copies of $D$) in each copy of $W^4$. Since $S^4 - (D' + D'') \approx (W^4 - D') \cup (W^4 - D'') \approx (Bd W^4 \times [0, 1]) \cup (Bd W^4 \times [0, 1])$ and $\pi_1(Bd W^4) \neq 1$, we have $\pi_1(S^4 - (D' + D'')) \neq 1$. Let $A$ be a polyhedral arc in $S^4$ such that $A \cap D' = a'$, $A \cap D'' = a''$ (where $a'$, $a''$ correspond to $a (= b = c)$ in $D)$ and $A \cap Bd W^4 = \{p\}$. Such an $A$ can easily be gotten because of the particular embedding of $D'$, $D''$ in each copy of $W^4$. Then $K_2 = D' \cup A \cup D''$ will be an embedding of $K$ in $S^4$ such that $\pi_1(S^4 - K_2) \neq 1$.

Finally, it is clear that the embeddings of $K_1$ and $K_2$ in $S^4$ are inequivalent since the fundamental groups of their complements are different.

Theorem 3. If $N_2$ is the regular neighborhood of $K_2$ in $S^4$ then $N_2 \times I \approx I^5$.

Proof of Theorem 3. Since $K_2 \approx D \cup A \cup D$, two disjoint copies of $D$ joined together by a polyhedra arc intersecting each $D$ in the point $a$ and $D \times I \setminus \{a\}$ (Theorem 1 [13]), it follows easily that $K_2 \times I \setminus 0$. Hence $N_2 \times I \setminus K_2 \times I \setminus 0$ and this implies that $N_2 \times I$ is a combinatorial 5-ball (Corollary 1, [12]).

Theorem 4. Suppose $K$ is a contractible 2-complex such that $K \times I \setminus 0$ and $K$ is embedded in the interior of a contractible 4-manifold with boundary $W^4 \subset E^4$ such that $\pi_1(W^4 - K) \neq 1$. Then given any combinatorial triangulation $T$ of $E^4$ which contains $K$ as a subcomplex, there exists no 3-manifold (with or without boundary) in $E^4$ which is a subcomplex of $T$ containing $K$.

Remark. Mazur's embedding of $D$ in $S^4$ is such a contractible 2-complex. It follows from the theorem that even though $D$ can be em-
bedded in $E^4$, for this particular embedding it lies in no 3-manifold in $E^4$.

Proof of Theorem 4. Suppose there exists such a 3-manifold $M^3$, that is $K \subset M^3 \subset T$. Then for some subdivision of $T$ and hence of $M^3$, say $\tilde{T}$, we would have $N(K, M^3) \subset \text{int } W^4$, where $N(K, M^3)$ denotes the simplicial neighborhood of $K$ in $M^3$ under the second barycentric subdivision of $\tilde{T}(M^3)$. Also let us suppose that $\tilde{T}$ is so fine that $N(N(K, M^3), \tilde{T}) \subset \text{int } W^4$. Now if $N(K, M^3) = I^3$, then $N(N(K, M^3), \tilde{T}) = I^4 \subset \text{int } W^4$. We then could use $\text{Bd } I^4 = S^3$ to shrink nontrivial curves of $W^4 - K$ missing $K$. (Also see Theorem 6 of [13].) Therefore, $N(K, M^3) \neq I^3$. However, $N(K, M^3) \times I \times K \times I \times 0$ and this implies that $N \times I = I^4$ which in turn implies $N = I^2([1], [2])$ which contradicts the above. This contradiction arose by assuming there existed an $M^3$ with $K \subset M^3 \subset T$.

4. Contractible complexes in $S^n$. If $K$ is a $k$-complex of a combinatorial $n$-sphere $S^n$, we will use $N(K, S^n)$ to denote the canonical regular neighborhood of $K$ under the second barycentric subdivision of $S^n$. $\Sigma K$ and $CK$ will denote the suspension of $K$ and cone over $K$ respectively. Also, we will write $\Sigma K = C^+K \cup C^-K$ with $C^+K \cap C^-K = K$, where in letting $p$ and $q$ denote the “top” and “bottom” points of $\Sigma K$ used in getting the suspension of $K$, we have that $C^+K$ is the cone over $K$ in $\Sigma K$ from $p$ and $C^-K$ is the cone over $K$ in $\Sigma K$ from $q$.

Lemma 3. Suppose $K$ is a $k$-complex in $S^n$ such that $N(K, S^n) \approx I^n$ and $B^n$ is a combinatorial $n$-ball in $S^n$ such that $N(\hat{K}, S^n) \subset \text{int } B^n$. If $\Sigma \hat{K} = K$ is considered as embedded in $S^{n+1} = B^{n+1} \cup (\text{Bd } B^{n+1})$, where $B^{n+1} = \Sigma B^n$, then $N(K, S^{n+1}) \approx I^{n+1}$ and $n_1(S^{n+1} - K) = 1$.

Proof of Lemma 3. $\Sigma [N(\hat{K}, S^n)]$ is a regular neighborhood of $K$ in $S^{n+1}$. That is, $\Sigma [N(\hat{K}, S^n)] \setminus \hat{K} = K$ since $N(\hat{K}, S^n) \setminus \hat{K}$ and it is an $n$-manifold with boundary since $\Sigma I^n \approx I^{n+1}$. Hence $I^{n+1} \approx \Sigma [N(\hat{K}, S^n)] = N(K, S^{n+1})$ (Theorem 23.1 [12]). It follows that $n_1(S^{n+1} - K) = 1$ since $K$ is cellular in $S^{n+1}$ (the decreasing sequence of $(n+1)$-cells are the canonical regular neighborhoods of $K$ under increasingly higher order barycentric subdivisions of $S^{n+1}$). That is $S^{n+1} - K = E^{n+1}$ [2].

Lemma 4. Suppose $K$ is a $k$-complex in $S^n$ ($n \geq 3$) such that $n_1(S^n - \hat{K}) \neq 1$ and $B^n$ is a combinatorial $n$-ball in $S^n$ such that $\hat{K} \subset \text{int } B^n$. Then if $\Sigma \hat{K} = K$ is considered as embedded in $S^{n+1}$ as in Lemma 3, then $n_1(S^{n+1} - K) \neq 1$.

Proof of Lemma 4. Since $n_1(S^n - \hat{K}) \neq 1$, we have that $n_1(B^n - \hat{K}) \neq 1$. Also $\Sigma B^n - \Sigma \hat{K} = B^{n+1} - K \approx (B^n - \hat{K}) \times (-1, 1)$. Hence
\[ \pi_1(B^{n+1} - K) \neq 1. \] The claim is that \( \pi_1(S^{n+1} - k) \neq 1. \) Suppose otherwise. Let \( J \) be any polyhedral simple closed curve in \( B^{n+1} - K \) which is nontrivial in \( B^{n+1} - K. \) Suppose \( J \) bounds a polyhedral singular disk \( D \) in \( S^{n+1} - K. \) Let \( p, q \) be the suspension points of \( \Sigma B^n \) and \( r \) the vertex point in \( C(Bd(\Sigma B^n)). \) Since \( n + 1 \geq 4, \) we can adjust \( D \) to a singular disk \( D' \) (keeping \( J \) fixed) so that \( D' \cap (\text{polyhedral arc } prq) = \emptyset. \) But then \( D' \) can be retracted onto a singular disk \( D'' \) bounded by \( J \) in \( B^{n+1} - K \) by projecting the part of \( D' \) not in \( B^{n+1} \) from \( r \) onto \( Bd B^{n+1} - \{p + q\}. \) This leads to a contradiction that \( \pi_1(B^{n+1} - K) \neq 1, \) therefore \( \pi_1(S^{n+1} - K) \neq 1. \)

**Lemma 5.** If \( K \) is a \( k \)-complex in \( S^n \) and \( \pi_1(S^n - K) \neq 1, \) denoting \( N(K, S^n) \) by \( N, \) then \( N \neq I^n, \) \( \pi_1(N - K) = \pi_1(Bd N) \neq 1 \) and \( \pi_1(Cl(S^n - N)) \neq 1. \)

**Proof of Lemma 5.** If \( N = I^n \) then \( K \) is cellular in \( S^n \) and this would imply that \( \pi_1(S^n - K) = 1, \) contradicting the hypothesis of the lemma. Also, \( S^n - K = (N - K) \cup Cl(S^n - N) \approx ([0, 1) \times Bd N) \cup Cl(S^n - N) \) (by Lemma 1). Hence \( S^n - K \) is homotopically equivalent to \( Cl(S^n - N). \) Therefore \( \pi_1(Cl(S^n - N)) \neq 1. \)

Suppose \( \pi_1(Bd N) = 1. \) Since \( S^n = N \cup Cl(S^n - N) \) and \( N \cap Cl(S^n - N) = Bd N, \) if \( \pi_1(Bd N) = 1, \) then using van Kampen's theorem we get that \( \pi_1(S^n) \) is the free product \( \pi_1(N) \ast \pi_1(S^n - N), \) which would not be trivial (Corollary 6.4.5, p. 244, [6]). Therefore, \( \pi_1(Bd N) \neq 1 \) and by Lemma 2 \( \pi_1(Bd N) = \pi_1(N - K) \neq 1. \)

**Lemma 6.** Suppose \( \hat{K} \) is a \( k \)-complex in \( S^n \) such that \( \hat{K} \times I \neq 0. \) Let \( K = \Sigma \hat{K} \) be p.w.l. embedded in \( S^{n+1} \) (not necessarily as in Lemma 3), then \( N(K, S^{n+1}) \times I \approx I^{n+1}. \)

**Proof of Lemma 6.** First we note that if \( \hat{L} \) is a subcomplex of \( \hat{K} \) such that \( \hat{K} \cap \hat{L} \neq \emptyset, \) \( K = \Sigma \hat{K} \) and \( L = \Sigma \hat{L} \) then \( K \cap L. \) This follows by induction on the number of simplexes of \( K - L. \) Next we observe that if \( \hat{K} \) is a complex such that \( \hat{K} \times I \neq 0 \) then \( k = \Sigma \hat{K} \times I \neq 0. \) This follows since \( \hat{K} \times \{v\} \) (\( v \) some vertex of \( \hat{K} \)) and by the above remark \( k = \Sigma \hat{K} \times \Sigma \times v \times v. \) Finally, if \( \hat{K} \) is a complex such that \( \hat{K} \times I \neq 0 \) and \( k = \Sigma \hat{K} \), then \( K \times I \neq 0. \) This follows since \( \Sigma \hat{K} \times I \neq \Sigma (\hat{K} \times I) \times I \) by the second remark.

Therefore since \( \hat{K} \times I \neq 0, \) we have that \( K \times I \neq 0. \) Hence, \( N(K, S^{n+1}) \times I \neq K \times I \neq 0. \) Therefore, \( N(K, S^{n+1}) \times I \neq I^{n+1}. \)

**Theorem 5.** For \( n \geq 4 \) there exists a contractible \((n - 2)\)-complex \( P \) and two inequivalent embeddings \( P_1, P_2 \) of \( P \) in \( S^n \) such that the regular neighborhood \( N_1 \) of \( P_1 \) is a combinatorial \( n \)-ball and \( \pi_1(S^n - P_1) = 1. \) However, \( \pi_1(S^n - P_2) \neq 1 \) and if \( N_2 \) is the regular neighborhood of \( P_2, \)
$N_2 \neq I^*$, $\pi_1(\text{Bd } N_2) = \pi_1(N_2 - P_2) \neq 1$ and $N_2 \times I$ is a combinatorial $(n+1)$-ball. Moreover $P \times I \setminus 0$.

**Proof of Theorem 5.** The proof will be by induction. For $n = 4$ the result follows from Theorem 2, Lemma 5, and Theorem 3. Suppose inductively for $n = k$ we have a contractible $(k-2)$-complex $P^{k-2}$, two embeddings $P^k_1, P^k_2$ in $S^k$ such that $N^k_1 \approx I^k$ and $\pi_1(S^k - P^k_{2-2}) = 1$, while $\pi_1(S^k - P^k_{2-2}) \neq 1$, $N^k_2 \neq I^k$, $\pi_1(\text{Bd } N^k_2) = \pi_1(N^k_2 - P^k_{2-2}) \neq 1$ and $N^k_2 \times I \approx I^{k+1}$. Also assume $P^{k+1} \times I \setminus 0$.

Using Lemma 3 we get a contractible $(k-1)$-complex $P^{k-1}_1 \approx \Sigma P^k_1$ in $S^{k+1}$ such that $N(P^{k-1}_1, S^{k+1}) \approx I^{k+1}$ and $\pi_1(S^{k+1} - P^{k-1}_1) = 1$. Using Lemma 4 we get a contractible $(k-1)$-complex $P^{k-1}_2 = \Sigma P^k_2$ in $S^{k+1}$ such that $\pi_1(S^{k+1} - P^{k-1}_2) \neq 1$. Lemma 5 then implies that $N^k_2 \approx I^k$, $\pi_1(\text{Bd } N^k_2) \neq 1$. Since $P^k_2 \approx \Sigma P^k_2$ and $P^k_2 \times I \setminus 0$, the third remark in the proof of Lemma 6 gives us that $P^k_2 \times I \setminus 0$. Also, Lemma 6 gives us that $N(P^k_2, S^{k+1}) \times I \approx I^{k+1}$. Finally, since $P^k_1 \times P^k_2$ and $P^k_1 \times P^k_2 \approx \Sigma P^k_2$ $(i = 1, 2)$ we have that $P^{k+1} \times I \setminus 0$.

**Corollary 6.** For $n \geq 4$ there exists a contractible $(n-1)$-complex $K^{n-1}$ in $S^n$ such that $N(K, S^n) \neq I^n$, $\pi_1(\text{Bd } N(K, S^n)) \neq 1$ and $N(K, S^n) \times I \approx I^{n+1}$. Also $\pi_1(S^n - K^{n-1}) \neq 1$.

**Corollary 7.** For $n \geq 4$ there exists a contractible $n$-complex (combinatorial $n$-manifold with boundary) $N^n$ in $S^n$ such that $N^n \neq I^n$, $\pi_1(\text{Bd } N^n) \neq 1$ and $N^n \times I \approx I^{n+1}$. Also $\pi_1(S^n - N^n) \neq 1$.

Corollary 7 follows from Theorem 5 by taking $N^n = N_2$ of that theorem; Corollary 6 by reducing $N^n$ to $K^{n-1}$ using Whitehead elementary contractions and the fact that $N(K^{n-1}, S^n) \approx N^n$, $\pi_1(S^n - N^n) \neq 1$ since $\pi_1(\text{Cl}(S^n - N^n)) \neq 1$ by Lemma 5. $\pi_1(S^n - K^{n-1}) \neq 1$ since we can assume that $K^{n-1} \subset \text{int } N^n$ and hence $S^n - K^{n-1}$ is of the same homotopy type as $\text{Cl}(S^n - N^n)$ (using Lemma 1).

**Theorem 6.** For $n \geq 5$, $N^n_2$ (of Theorem 5) is a contractible combinatorial $n$-manifold with boundary which is not topological $I^n$, but is combinatorially equivalent to the union of two combinatorial $n$-balls, $B^n_1 \cup B^n_2$ such that $B^n_1 \cap B^n_2 \approx B^n_2$ a combinatorial $n$-ball which is a subcomplex of each. Furthermore, int $N_2^n \approx X \cup Y$ where $X \approx Y \approx X \cap Y \approx E^n$, while int $N_2^n \neq E^n$.

**Proof of Theorem 6.** For $n \geq 5$, $N_2^n \approx N(P_2^{n-2}, S^n) \approx N(\Sigma P_2^{n-3}, S^n)$. Also, $N_2^n = N(P_2^{n-3}, S^{n-1}) \approx I^{n-1}$ with $N_2^{n-2} \times I \approx I^n$. We observe that $N_2^n \approx N(C + P_2^{n-3}, S^n) \cup N(C - P_2^{n-3}, S^n)$. Since $C + P_2^{n-3} \approx 0$ and $C - P_2^{n-3} \approx 0$, Theorem 23, [12] gives us that $N(C + P_2^{n-3}, S^n) \approx B^n_1$ and
$N(C-P_2^{n-3}, S^n) \approx B_3^n$, the two desired combinatorial $n$-balls. Since $S^n$ was obtained as $\Sigma B^{n-1} \cup C(Bd \Sigma B^{n-1})$ where $N_2^{n-1} = N(P_2^{n-3}, S^{n-1})$ lies in int $B_2^{n-1}$, for some $B_2^{n-1} \subset S^{n-1}$, it follows that $B_4^n \cap B_2^n \approx N(C+P_2^{n-3}, S^n) \setminus N(C-P_2^{n-3}, S^n) \approx N(P_2^{n-3}, S^n) \approx N(P_2^{n-3}, S^{n-1}) \times I$ which is $\approx I^n$, that is, our $B_3^n$. $N(P_2^{n-3}, S^n) \approx N(P_2^{n-3}, S^{n-1}) \times I$ since the latter expression is clearly a regular neighborhood of $P_2^{n-3}$ in $S^n$ and any two regular neighborhoods of the same complex are combinatorially equivalent.

Letting $X = \text{int } B_1^n$, $Y = \text{int } B_2^n$, then $X \cap Y \approx \text{int } B_3^n$ so that int $B_2^n \approx X \cup Y$ and $X \approx Y \approx X \cap Y \approx E^n$. We have that int $N_2^n \neq E^n$ since $N_2^n$ is an $n$-manifold with boundary (and hence collared from the inside [3]) and $\pi_1(Bd N_2^n) \neq 1$. That is, if int $N_2^n = E^n$, simple closed curves near “infinity” can be shrunk near “infinity,” but Bd $N_2^n \times (0, 1)$ the collar of Bd $N_2^n$ in $N_2^n$ is not simply connected and hence there exist nontrivial simple closed curves in Bd $N_2^n \times (0, 1)$.

**Theorem 7.** Suppose $C'$ is a contractible $k$-complex that can be p.w.l. embedded in a combinatorial $n$-sphere $S^n$ with triangulation $T$ as a subcomplex $C$ such that $\pi_1(S^n-C) \neq 1$ (necessarily $k = n$, $n-1$, or $n-2$ by Lemma 5 and Theorem 1 [5]). Then for $n \geq 5$ there exists a p.w.l. embedding $\hat{C}$ of $C'$ in $S^n$ under $T$ such that $N(\hat{C}, S^n) \approx N(C, S^n)(\neq I^n)$, but now $\pi_1(S^n-\hat{C}) = 1$.

**Proof of Theorem 7.** Let $\Sigma$ be the combinatorial $n$-manifold formed by attaching two copies of $N(C, S^n)$ together along their boundaries. Since $N(C, S^n)$ is contractible, $\Sigma$ is a combinatorial $n$-manifold with the homotopy type of $S^n$. Hence for $n \geq 5$, $\Sigma$ is a topological $n$-sphere which is also a combinatorial $n$-manifold [10], [14]. Let us also denote $C$ in $\Sigma$ as the complex $C$ in one copy of $N(C, S^n)$ used in forming $\Sigma$. Now $\pi_1(\Sigma - C) = 1$ since $\Sigma - C = \{0, 1\} \times Bd N(C, S^n) \cup N(C, S^n)$ which is homotopically equivalent to $N(C, S^n)$. Let $\mathfrak{p}$ be an interior point of some $n$-simplex of $\Sigma$ missing the copy of $N(C, S^n)$ in $\Sigma$ containing $C$. Now $\Sigma - \{\mathfrak{p}\}$ is p.w.l. equivalent to $S^n - \{q\}$ under $T$ for some $q \in S^n$ since $n \geq 5$ [11]. Hence there exists a p.w.l. homeomorphism $h$ of $\Sigma - \{\mathfrak{p}\}$ onto $S^n - \{q\}$ taking $C$ and $N(C, S^n)$ (as in $\Sigma - \{\mathfrak{p}\}$) into $S^n - \{q\}$ (under $T$). Then $h(C) = \hat{C}$ is a p.w.l. embedding of $C'$ in $S^n$ and $\pi_1(S^n-\hat{C}) = 1$ since $\pi_1(\Sigma - C) = 1$. Since $h(N(C, S^n))$ is a regular neighborhood of $\hat{C}$ in $S^n$ under a subdivision of $T$, $N(\hat{C}, S^n) \approx h(N(C, S^n)) \approx N(C, S^n)$. Note, if $C'$ is the contractible $k$-complex given in Theorem 5, then one has that $\Sigma$ is in fact a combinatorial $n$-sphere (since $N \times I \approx I^{n+1}$). Hence $\Sigma \approx S^n$ under $T$ and the result follows immediately.
BIBLIOGRAPHY


5. ———, *Regular neighborhoods*, mimeographed notes, Florida State University, pp. 1–21.


