CONTRACTIBLE COMPLEXES IN $S^n$

LESLEY C. GLASER

1. Introduction. By a pseudo $n$-cell is meant a contractible compact combinatorial $n$-manifold with boundary. Poenaru [9] and Mazur [7] gave the first examples of pseudo 4-cells which are not topological 4-cells, but whose products with the unit interval are topologically 5-cells. Newman [8] defines a 2-complex $P$ such that $\pi_1(P) \neq 1$, while $H_1(P, Z) = 0 = H_2(P, Z)$. Curtis [4] making use of this 2-complex has shown that, for each $n \geq 4$, there exists a pseudo $n$-cell which is not a topological $n$-cell because its boundary fails to be simply connected. Curtis [4] also shows that the cartesian product of a pseudo $n$-cell and an interval is the topological $(n+1)$-cell, $I^{n+1}$ if $n \geq 5$.

Curtis [5] making use of Mazur's peculiar embedding of the dunce hat in $S^4$ [7], [13] gives an example of a contractible 2-complex $K$ embedded as a subcomplex of a combinatorial triangulation of $S^4$ such that $\pi_1(S^4 - K) \neq 1$. The purpose of this paper is to show that for $n \geq 4$ there exists a contractible $(n-2)$-complex $K^{n-2}$ combinatorially embedded in $S^n$ such that $\pi_1(S^n - K^{n-2}) \neq 1$. The regular neighborhood $N^n = N(K^{n-2})$ of $K^{n-2}$ in $S^n$ is also a pseudo $n$-cell which fails to be a topological $n$-cell and its product with the unit interval $I$ is shown to be a combinatorial $(n+1)$-cell, rather than just merely topological. In addition, each $N^n (n \geq 5)$ gives examples of combinatorial $n$-manifolds with boundary which are not topologically $I^n$ but can be expressed as the union of two combinatorial $n$-balls whose intersection is also a combinatorial $n$-ball.

2. Definitions. We will use the terminology of [12], [13]. All manifolds and all mappings or homeomorphisms will be considered in the combinatorial sense. We will use $\approx$ to denote combinatorial equivalence. If the complex $K$ collapses to the complex $L$, this will be denoted $K \simeq L$.

Let $f: X \rightarrow Y$ be continuous. The identification space $Y_f$ derived from $(X \times [0, 1]) \cup Y$ by identifying each point $(x, 1)$ with the point

---

1 This paper is the substance of the second half of the author's Ph.D. thesis which was prepared under the supervision of Professor R. H. Bing at the University of Wisconsin.

Presented to the Society, August 27, 1964; received by the editors January 22, 1965.

1357
f(x) in Y and using the identification topology is called the mapping cylinder of f.

3. Preliminaries. The following two lemmas are well known and elementary, hence no proof will be included.

**Lemma 1.** If $C$ is a $k$-complex embedded as a finite subcomplex of a combinatorial $n$-sphere $S^n$ and $M$ is a regular neighborhood of $C$ in $S$, with $C \subset \text{int} M$, then there is a combinatorial map $\phi: \text{Bd } M \to C$ such that $M$ is combinatorially equivalent to $I \times \text{Bd } M \cup \text{Cyl } \phi$, the mapping cylinder of $\phi$.

**Lemma 2.** Suppose $C$ is a $k$-complex embedded as a finite subcomplex of a combinatorial $n$-sphere $S^n$ and $N$ is any regular neighborhood of $C$ in $S^n$, such that $C \subset \text{int } N$; then $\pi_1(N-C) = \pi_1(\text{Bd } N)$.

The topological dunce hat $D$ is obtained from a triangle $abc$ say, by identifying all three sides $ab = bc = ac$.

**Theorem 1.** There exist two combinatorially inequivalent embeddings $D_1, D_2$ of the dunce hat $D$ in $S^4$, such that the regular neighborhood $N_1$ of $D_1$ is combinatorially $I^4$, while $N_2$ the regular neighborhood of $D_2$ is not topologically $I^4$. Moreover, $\pi_1(\text{Bd } N_2) \neq 1$, $\pi_1(N_2-D_2) \neq 1$, but $\pi_1(S^4-D_2) = 1$.

**Proof of Theorem 1.** Let $D_1$ be any combinatorial embedding of $D$ in $S^4 \subset S^4$. Then $N_1 \subset N_1$, the regular neighborhood of $D$ in $S^4$, and since $N_1 \approx I^4$, $N_1 \approx I^4$.

For $D_2$ we will use Mazur's embedding of $D$ in $S^4$ (as in Theorem 5 [13]). Since $N_2 \approx W^4$ (also Theorem 5 [13]) and $\pi_1(\text{Bd } W^4) \neq 1$ (see [7]) we have that $N_2 \neq I^4$. The fact that $\pi_1(N_2-D_2) \neq 1$ follows from Lemma 2. We see that $\pi_1(S^4-D_2) = 1$ by considering Mazur's embedding of $D$ in $S^4$. That is $D \subset W^4 \subset W^4 \approx S^4$. Since $S^4-D_2 \approx W^4 \cup W^4-D_2 \approx W^4 \cup (\text{Bd } W \times [0,1])$ (using Lemma 1), we see that $S^4-D_2$ is of the same homotopy type as $W^4$ and $\pi_1(S^4-D_2) = 1$.

To see that these two embeddings are combinatorially inequivalent, suppose there exists a p.w.l. homeomorphism taking $S^4$ onto $S^4$ carrying $D_1$ onto $D_2$. Let $a_1, a_2$ be the points of $D_1, D_2$ respectively, which correspond to the point $a(=b=c)$ in $D$. Then by subdividing the triangulation of $S^4$ so that $h$ is simplicial, we get that $h$ carries $\text{st}(a_1, S^4)$ onto $\text{st}(a_2, S^4)$, each combinatorial 4-balls. Also $h$ carries $\text{lk}(a_1, D_1) \subset \text{lk}(a_1, S^4) \approx S^4$ onto $\text{lk}(a_2, D_2) \subset \text{lk}(a_2, S^4) \approx S^4$. This leads to a contradiction, since there exists no homeomorphism of $S^4$ onto $S^4$ carrying $\text{lk}(a_1, D_1)$ as in $\text{lk}(a_1, S^4)$ onto $\text{lk}(a_2, D_2)$ as in $\text{lk}(a_2, S^4)$. See Figures 5 and 8 of [13].
Theorem 2. There exists a contractible 2-complex $K$ and two inequivalent embeddings $K_1$, $K_2$ of $K$ in $S^4$ so that the regular neighborhood $N_1$ of $K_1$ is a combinatorial 4-ball, while $\pi_1(S^4 - K_2) \neq 1$.

Remark. Since $N_1 \cong I^4$, $K_1$ is cellular in $S^4$ and hence $S^4 - K_1 = E^4$ and $\pi_1(S^4 - K_1) = 1$. Also it will follow from a later result, which does not use the particular construction of the embedding of $K_2$ in $S^4$, that if $N_2$ is the regular neighborhood of $K_2$ in $S^4$ then $\pi_1(\text{Bd} N_2) \neq 1$ and hence $N_2 \neq I^4$.

Proof of Theorem 2. $K$ will be the union of two disjoint copies of the dunce hat $D$ joined together by a polyhedral segment intersecting each in $a(=b=c)$. $K_1$ will be the embedding of $K$ in $S^4 \subset S^4$ and $N_1 \cong I^4$ as in Theorem 1.

To get $K_2$, we will use Curtis's modification [5]. Let us again consider $S^4$ as $2W^4$ (Mazur's pseudo 4-cell). We have a $D'$ and $D''$ (copies of $D$) in each copy of $W^4$. Since $S^4 - (D' + D'') \cong (W^4 - D') \cup (W^4 - D'') \cong (\text{Bd} W^4 \times [0, 1]) \cup (\text{Bd} W^4 \times [0, 1])$ and $\pi_1(\text{Bd} W^4) \neq 1$, we have $\pi_1(S^4 - (D' + D'')) \neq 1$. Let $A$ be a polyhedral arc in $S^4$ such that $A \cap D' = a'$, $A \cap D'' = a''$ (where $a'$, $a''$ correspond to $a(=b=c)$ in $D$) and $A \cap \text{Bd} W^4 = \{p\}$. Such an $A$ can easily be gotten because of the particular embedding of $D'$, $D''$ in each copy of $W^4$. Then $K_2 = D' \cup A \cup D''$ will be an embedding of $K$ in $S^4$ such that $\pi_1(S^4 - K_2) \neq 1$.

Finally, it is clear that the embeddings of $K_1$ and $K_2$ in $S^4$ are inequivalent since the fundamental groups of their complements are different.

Theorem 3. If $N_2$ is the regular neighborhood of $K_2$ in $S^4$ then $N_2 \times I \cong I^5$.

Proof of Theorem 3. Since $K_2 \cong D \cup A \cup D$, two disjoint copies of $D$ joined together by a polyhedra arc intersecting each $D$ in the point $a$ and $D \times I \setminus \{a\}$ (Theorem 1 [13]), it follows easily that $K_2 \times I \setminus 0$. Hence $N_2 \times I \setminus K \times I \setminus 0$ and this implies that $N_2 \times I$ is a combinatorial 5-ball (Corollary 1 [12]).

Theorem 4. Suppose $K$ is a contractible 2-complex such that $K \times I \setminus 0$ and $K$ is embedded in the interior of a contractible 4-manifold with boundary $W^4 \subset E^4$ such that $\pi_1(W^4 - K) \neq 1$. Then given any combinatorial triangulation $T$ of $E^4$ which contains $K$ as a subcomplex, there exists no 3-manifold (with or without boundary) in $E^4$ which is a subcomplex of $T$ containing $K$.

Remark. Mazur's embedding of $D$ in $S^4$ is such a contractible 2-complex. It follows from the theorem that even though $D$ can be em-
bedded in $E^4$, for this particular embedding it lies in no 3-manifold in $E^4$.

**Proof of Theorem 4.** Suppose there exists such a 3-manifold $M^3$, that is $K \subset M^3 \subset T$. Then for some subdivision of $T$ and hence of $M^3$, say $\hat{T}$, we would have $N(K, M^3) \subset \text{int } W^4$, where $N(K, M^3)$ denotes the simplicial neighborhood of $K$ in $M^3$ under the second barycentric subdivision of $\hat{T}(M^3)$. Also let us suppose that $\hat{T}$ is so fine that $N(N(K, M^3), \hat{T}) \subset \text{int } W^4$. Now if $N(K, M^3) = I^3$, then $N(N(K, M^3), \hat{T}) = I^4 \subset \text{int } W^4$. We then could use $\text{Bd } I^4 = S^3$ to shrink nontrivial curves of $W^4 - K$ missing $K$. (Also see Theorem 6 of [13].) Therefore, $N(K, M^3) \neq I^3$. However, $N(K, M^3) \times I \times I \times I \times 0$ and this implies that $N \times I = I^4$ which in turn implies $N = I^4([1], [2])$ which contradicts the above. This contradiction arose by assuming there existed an $M^3$ with $K \subset M^3 \subset T$.

4. **Contractible complexes in $S^n$.** If $K$ is a $k$-complex of a combinatorial $n$-sphere $S^n$, we will use $N(K, S^n)$ to denote the canonical regular neighborhood of $K$ under the second barycentric subdivision of $S^n$. $\Sigma K$ and $CK$ will denote the suspension of $K$ and cone over $K$ respectively. Also, we will write $\Sigma K = C^+K \cup C^-K$ with $C^+K \cap C^-K = K$, where in letting $p$ and $q$ denote the “top” and “bottom” points of $\Sigma K$ used in getting the suspension of $K$, we have that $C^+K$ is the cone over $K$ in $\Sigma K$ from $p$ and $C^-K$ is the cone over $K$ in $\Sigma K$ from $q$.

**Lemma 3.** Suppose $\hat{K}$ is a $k$-complex in $S^n$ such that $N(\hat{K}, S^n) \approx I^n$ and $B^n$ is a combinatorial $n$-ball in $S^n$ such that $N(\hat{K}, S^n) \subset \text{int } B^n$. If $\Sigma \hat{K} = K$ is considered as embedded in $S^{n+1} = B^{n+1} \cup (\text{Bd } B^{n+1})$, where $B^{n+1} = \Sigma B^n$, then $N(K, S^{n+1}) \approx I^{n+1}$ and $\pi_1(S^{n+1} - K) = 1$.

**Proof of Lemma 3.** $\Sigma [N(\hat{K}, S^n)]$ is a regular neighborhood of $K$ in $S^{n+1}$. That is, $\Sigma [N(\hat{K}, S^n)] \setminus \Sigma \hat{K} = K$ since $N(\hat{K}, S^n) \setminus \hat{K}$ and it is an $n$-manifold with boundary since $\Sigma I^n \approx I^{n+1}$. Hence $I^{n+1} \approx \Sigma [N(\hat{K}, S^n)] = N(K, S^{n+1})$ (Theorem 23, [12]). It follows that $\pi_1(S^{n+1} - K) = 1$ since $K$ is cellular in $S^{n+1}$ (the decreasing sequence of $(n+1)$-cells are the canonical regular neighborhoods of $K$ under increasingly higher order barycentric subdivisions of $S^{n+1}$). That is $S^{n+1} - K = E^{n+1}$ [2].

**Lemma 4.** Suppose $\hat{K}$ is a $k$-complex in $S^n (n \geq 3)$ such that $\pi_1(S^n - \hat{K}) \neq 1$ and $B^n$ is a combinatorial $n$-ball in $S^n$ such that $\hat{K} \subset \text{int } B^n$. Then if $\Sigma \hat{K} = K$ is considered as embedded in $S^{n+1}$ as in Lemma 3, then $\pi_1(S^{n+1} - K) \neq 1$.

**Proof of Lemma 4.** Since $\pi_1(S^n - \hat{K}) \neq 1$, we have that $\pi_1(B^n - \hat{K}) \neq 1$. Also $\Sigma B^n = \Sigma \hat{K} = B^{n+1} - K \approx (B^n - \hat{K}) \times (-1, 1)$. Hence
The claim is that $\pi_1(B^{n+1} - K) \neq 1$. Suppose otherwise. Let $J$ be any polyhedral simple closed curve in $B^{n+1} - K$ which is nontrivial in $B^{n+1} - K$. Suppose $J$ bounds a polyhedral singular disk $D$ in $S^{n+1} - K$. Let $p, q$ be the suspension points of $\Sigma B^n$ and $r$ the vertex point in $C(Bd(\Sigma B^n))$. Since $n + 1 \geq 4$, we can adjust $D$ to a singular disk $D'$ (keeping $J$ fixed) so that $D' \cap (\text{polyhedral arc } pq) = \emptyset$. But then $D'$ can be retracted onto a singular disk $D''$ bounded by $J$ in $B^{n+1} - K$ by projecting the part of $D'$ not in $B^{n+1}$ from $r$ onto $Bd B^{n+1} - \{p + q\}$. This leads to a contradiction that $\pi_1(B^{n+1} - K) \neq 1$, therefore $\pi_1(B^{n+1} - K) \neq 1$.

**Lemma 5.** If $K$ is a $k$-complex in $S^n$ and $\pi_1(S^n - K) \neq 1$, denoting $N(K, S^n)$ by $N$, then $N \neq I^n$, $\pi_1(N - K) = \pi_1(Bd N) \neq 1$ and $\pi_1(Cl(S^n - N)) \neq 1$.

**Proof of Lemma 5.** If $N = I^n$ then $K$ is cellular in $S^n$ and this would imply that $\pi_1(S^n - K) = 1$, contradicting the hypothesis of the lemma. Also, $S^n - K = (N - K) \cup Cl(S^n - N) = ([0, 1) \times Bd N) \cup Cl(S^n - N)$ (by Lemma 1). Hence $S^n - K$ is homotopically equivalent to $Cl(S^n - N)$. Therefore $\pi_1(Cl(S^n - N)) \neq 1$.

Suppose $\pi_1(Bd N) = 1$. Since $S^n = N \cup Cl(S^n - N)$ and $N \cap Cl(S^n - N) = Bd N$, if $\pi_1(Bd N) = 1$, then using van Kampen’s theorem we get that $\pi_1(S^n)$ is the free product $\pi_1(N) * \pi_1(S^n - N)$, which would not be trivial (Corollary 6.4.5, p. 244, [6]). Therefore, $\pi_1(Bd N) \neq 1$ and by Lemma 2 $\pi_1(Bd N) = \pi_1(N - K) \neq 1$.

**Lemma 6.** Suppose $K$ is a $k$-complex in $S^n$ such that $K \times I \neq 0$. Let $K = \Sigma K$ be p.w.l. embedded in $S^{n+1}$ (not necessarily as in Lemma 3), then $N(K, S^{n+1}) \times I = I^{n+2}$.

**Proof of Lemma 6.** First we note that if $L$ is a subcomplex of $K$ such that $K \times L$, if $K = \Sigma K$ and $L = \Sigma L$ then $K \times L$. This follows by induction on the number of simplexes of $K - L$. Next we observe that if $K$ is a complex such that $K \times I \neq 0$ then $K = \Sigma K \times 0$. This follows since $K \times \{v\}$ (for some vertex of $K$) and by the above remark $K = \Sigma K \times 0 \times 0$. Finally, if $K$ is a complex such that $K \times I \neq 0$ and if $K = \Sigma K$, then $K \times I \neq 0$. This follows since $\Sigma K \times I \times Cl(K \times I)$ and $\Sigma (K \times I) \times I$ by the second remark.

Therefore since $K \times I \neq 0$, we have that $K \times I \neq 0$. Hence, $N(K, S^{n+1}) \times I \times K \times I \neq I^{n+2}$.

**Theorem 5.** For $n \geq 4$ there exists a contractible $(n - 2)$-complex $P$ and two inequivalent embeddings $P_1, P_2$ of $P$ in $S^n$ such that the regular neighborhood $N_1$ of $P_1$ is a combinatorial $n$-ball and $\pi_1(S^n - P_1) = 1$. However, $\pi_1(S^n - P_2) \neq 1$ and if $N_2$ is the regular neighborhood of $P_2$,
$N_2 \neq I^n$, $\pi_1(\text{Bd } N_2) = \pi_1(N_2 - P_2) \neq 1$ and $N_2 \times I$ is a combinatorial $(n+1)$-ball. Moreover $P \times I \subseteq 0$.

**Proof of Theorem 5.** The proof will be by induction. For $n = 4$ the result follows from Theorem 2, Lemma 5, and Theorem 3. Suppose inductively for $n = k$ we have a contractible $(k-2)$-complex $\mathcal{P}^{k-2}$, two embeddings $P_1^{k-2}, P_2^{k-2}$ in $S^k$ such that $N_1^k \approx I^k$ and $\pi_1(S^k - P_1^{k-2}) = 1$, while $\pi_1(S^k - P_2^{k-2}) \neq 1$, $N_2^k \neq I^k$, $\pi_1(\text{Bd } N_2^k) = \pi_1(N_2^k - P_2^{k-2}) \neq 1$ and $N_2^k \times I \approx I^{k+1}$. Also assume $P^{k+1} \times I \subseteq 0$.

Using Lemma 3 we get a contractible $(k-1)$-complex $P_1^{k-1} \approx \Sigma P_1^{k-2}$ in $S^{k+1}$ such that $N(P_1^{k-1}, S^{k+1}) \approx I^{k+1}$ and $\pi_1(S^{k+1} - P_1^{k-1}) = 1$. Using Lemma 4 we get a contractible $(k-1)$-complex $P_2^{k-1} \approx \Sigma P_2^{k-2}$ in $S^{k+1}$ such that $\pi_1(S^{k+1} - P_2^{k-1}) \neq 1$. Lemma 5 then implies that $N_2^{k+1} \neq I^{k+1}$, $\pi_1(N_2^{k+1} - P_2^{k-1}) = \pi_1(\text{Bd } N_2^{k+1}) \neq 1$. Since $P_2^{k-1} \approx \Sigma P_2^{k-2}$ and $P_2^{k-2} \times I \subseteq 0$, the third remark in the proof of Lemma 6 gives us that $P_2^{k-1} \times I \subseteq 0$. Also, Lemma 6 gives us that $N(P_2^{k-1}, S^{k+1}) \times I \approx I^{k+2}$. Finally, since $P_2^{k-2} \approx P_2^{k-2} + P_2^{k-1}$ and $P_2^{k-1} \approx \Sigma P_2^{k-2}$, we have that $P_2^{k-1} \approx P_2^{k-2}$.

**Corollary 6.** For $n \geq 4$ there exists a contractible $(n-1)$-complex $K^{n-1}$ in $S^n$ such that $N(K, S^n) \neq I^n$, $\pi_1(\text{Bd } N(K, S^n)) \neq 1$ and $N(K, S^n) \times I \approx I^{n+1}$. Also $\pi_1(S^n - K^{n-1}) \neq 1$.

**Corollary 7.** For $n \geq 4$ there exists a contractible n-complex (combinatorial n-manifold with boundary) $N^n$ in $S^n$ such that $N^n \neq I^n$, $\pi_1(\text{Bd } N^n) \neq 1$ and $N^n \times I \approx I^{n+1}$. Also $\pi_1(S^n - N^n) \neq 1$.

Corollary 7 follows from Theorem 5 by taking $N^n = N_2$ of that theorem; Corollary 6 by reducing $N^n$ to $K^{n-1}$ using Whitehead elementary contractions and the fact that $N(K^{n-1}, S^n) \approx N^n$, $\pi_1(S^n - N^n) \neq 1$ since $\pi_1(CI(S^n - N_2)) \neq 1$ by Lemma 5. $\pi_1(S^n - K^{n-1}) \neq 1$ since we can assume that $K^{n-1} \subseteq \text{int } N^n$ and hence $S^n - K^{n-1}$ is of the same homotopy type as $\text{CI}(S^n - N^n)$ (using Lemma 1).

**Theorem 6.** For $n \geq 5$, $N_2^n$ (of Theorem 5) is a contractible combinatorial n-manifold with boundary which is not topological $I^n$, but is combinatorially equivalent to the union of two combinatorial n-balls, $B_1^n \cup B_2^n$ such that $B_1^n \cap B_2^n \approx B_3^n$ a combinatorial n-ball which is a subcomplex of each. Furthermore, $\text{int } N_2^n \approx X \cup Y$ where $X \approx Y \approx X \cap Y \approx E^n$, while $\text{int } N_2^n \neq E^n$.

**Proof of Theorem 6.** For $n \geq 5$, $N_2^n \approx N(P_2^n, S^n) \approx N(\Sigma P_2^{n-3}, S^n)$. Also, $N_2^n = N(P_2^n - S^n) \neq I^{-1}$ with $P_2^n - S^n \times I \neq I^n$. We observe that $N_2^n \approx N(C + P_2^{n-3}, S^n) \cup N(C - P_2^{n-3}, S^n)$. Since $C + P_2^{n-3} \subseteq 0$ and $C - P_2^{n-3} \subseteq 0$, Theorem 23a [12] gives us that $N(C + P_2^{n-3}, S^n) \approx B_1^n$ and
\(N(C-P^{n-3}, S^n) \approx B^n_2\), the two desired combinatorial \(n\)-balls. Since \(S^n\) was obtained as \(\Sigma B^n-1 \cup (\text{Bd } \Sigma B^n-1)\) where \(N_2^n-1 = N(P_2^{n-3}, S^{n-1})\) lies in \(\text{int } B_2^n-1\), for some \(B_2^n \supset S^n-1\), it follows that \(B_2^n \cap B_2^n \approx N(C+P^{n-3}, S^n) \cap N(C-P^{n-3}, S^n) \approx N(P_2^{n-3}, S^n) \approx N(P_2^{n-3}, S^{n-1}) \times I\) which is \(\approx I^n\), that is, our \(B^n_2\). \(N(P_2^{n-3}, S^n) \approx N(P_2^{n-3}, S^{n-1}) \times I\) since the latter expression is clearly a regular neighborhood of \(P_2^{n-3}\) in \(S^n\) and any two regular neighborhoods of the same complex are combinatorially equivalent.

Letting \(X = \text{int } B^n_1, Y = \text{int } B^n_2\), then \(X \cap Y \approx \text{int } B^n_2\) so that \(\text{int } N^n_2 = X \cup Y\) and \(X \approx Y \approx X \cap Y \approx E^n\). We have that \(\text{int } N^n_2 \neq E^n\) since \(N^n_2\) is an \(n\)-manifold with boundary (and hence collared from the inside \([3]\)) and \(\pi_1(\text{Bd } N^n_2) \neq 1\). That is, if \(\text{int } N^n_2 = E^n\), simple closed curves near “infinity” can be shrunk near “infinity,” but \(\text{Bd } N^n_2 \times [0, 1)\) the collar of \(\text{Bd } N^n_2\) in \(N^n_2\) is not simply connected and hence there exist nontrivial simple closed curves in \(\text{Bd } N^n_2 \times (0, 1)\).

**Theorem 7.** Suppose \(C\) is a contractible \(k\)-complex that can be p.w.l. embedded in a combinatorial \(n\)-sphere \(S^n\) with triangulation \(T\) as a subcomplex \(C\) such that \(\pi_1(S^n- C) \neq 1\) (necessarily \(k = n, n-1\), or \(n-2\) by Lemma 5 and Theorem 1 \([5]\)). Then for \(n \geq 5\) there exists a p.w.l. embedding \(\hat{C}\) of \(C\) in \(S^n\) under \(T\) such that \(N(\hat{C}, S^n) \approx N(C, S^n) (\neq I^n)\), but now \(\pi_1(S^n- \hat{C}) = 1\).

**Proof of Theorem 7.** Let \(\Sigma\) be the combinatorial \(n\)-manifold formed by attaching two copies of \(N(C, S^n)\) together along their boundaries. Since \(N(C, S^n)\) is contractible, \(\Sigma\) is a combinatorial \(n\)-manifold with the homotopy type of \(S^n\). Hence for \(n \geq 5\), \(\Sigma\) is a topological \(n\)-sphere which is also a combinatorial \(n\)-manifold \([10], [14]\). Let us also denote \(C\) in \(\Sigma\) as the complex \(C\) in one copy of \(N(C, S^n)\) used in forming \(\Sigma\). Now \(\pi_1(\Sigma- C) = 1\) since \(\Sigma- C = \{0, 1\} \times \text{Bd } N(C, S^n)\) \(\cup N(C, S^n)\) which is homotopically equivalent to \(N(C, S^n)\). Let \(p\) be an interior point of some \(n\)-simplex of \(\Sigma\) missing the copy of \(N(C, S^n)\) in \(\Sigma\) containing \(C\). Now \(\Sigma - \{p\}\) is p.w.l. equivalent to \(S^n- \{q\}\) under \(T\) for some \(q \in S^n\) since \(n \geq 5\) \([11]\). Hence there exists a p.w.l. homeomorphism \(h\) of \(\Sigma - \{p\}\) onto \(S^n- \{q\}\) taking \(C\) and \(N(C, S^n)\) (as in \(\Sigma - \{p\}\) into \(S^n- \{q\}\) (under \(T\)). Then \(h(C) = \hat{C}\) is a p.w.l. embedding of \(C\) in \(S^n\) and \(\pi_1(S^n- \hat{C}) = 1\) since \(\pi_1(\Sigma-C) = 1\). Since \(h(N(C, S^n))\) is a regular neighborhood of \(\hat{C}\) in \(S^n\) under a subdivision of \(T, N(\hat{C}, S^n) \approx h(N(C, S^n)) \approx N(C, S^n)\). Note, if \(C\) is the contractible \(k\)-complex given in Theorem 5, then one has that \(\Sigma\) is in fact a combinatorial \(n\)-sphere (since \(N \times I \approx I^{n+1}\)). Hence \(\Sigma \approx S^n\) under \(T\) and the result follows immediately.
Bibliography

1. R. H. Bing, A set is a 3-cell if its cartesian product with an arc is a 4-cell, Proc. Amer. Math. Soc. 12 (1961), 13–19.


5. ———, Regular neighborhoods, mimeographed notes, Florida State University, pp. 1–21.


Rice University and

The University of Wisconsin