A PROBABILITY BOUND FOR INTEGRALS WITH RESPECT TO STOCHASTIC PROCESSES WITH INDEPENDENT INCREMENTS[1]

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1. **Introduction.** The purpose of this paper is to establish the following theorem.

THEOREM. *Let $X=\{X_t: -\infty<t<\infty\}$ be a real-valued stochastic process with independent increments. Let μ be a σ-finite measure on the real line $\mathcal{R}$ which has the property that for every $\beta>0$ there exists $T_\beta>0$ such that whenever $|\lambda|\leqq T_\beta$ and $-\infty<s\leqq t<\infty$,*

$$(1.1)\quad E\exp(\lambda[(X_t-X_s)-E(X_t-X_s)])\leqq \exp(\beta|\lambda|\mu(s,t]).$$

Then for every function $f\in L_1(\mu)\cap L_\infty(\mu)$ (over $\mathcal{R}$) for which $\|f\|_1\leqq 1$, the random variable $\int f(t)d[X_t-EX_t]$ is well defined as a limit-in-the-mean of order 2, and for every $\epsilon>0$ there exists a positive number $\rho<1$ (depending only on ϵ) such that

$$(1.2)\qquad P\left[\left|\int f(t)d[X_t-EX_t]\right|>\epsilon\right]\leqq 2\rho^{1/\|f\|_\infty}.$$

If $f(s,t)$ is a real-valued function on $\mathcal{R}\times\mathcal{R}$ such that

$$f(s,\cdot)\in L_1(\mu)\cap L_\infty(\mu),\quad \|f(s,\cdot)\|_1\leqq 1\quad\text{and}\quad \|f(s,\cdot)\|_\infty=1/\gamma(s)$$

then if the stochastic process $\{Y_s: -\infty<s<\infty\}$ is defined by

$$Y_s=\int f(s,t)d[X_t-EX_t],$$

it will follow as an immediate consequence of the Theorem that for every $\epsilon>0$ there exists $0<\rho<1$ such that

$$P[|Y_s|>\epsilon]\leqq 2\rho^{\gamma(s)}.$$

Thus, our Theorem provides a useful probability bound for a large class of stochastic processes derived from processes with independent increments. In particular, if s^* is held fixed and χ_s denotes the set characteristic function of the interval $(s^*, s]$, then taking

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$$f(s, t) = \frac{\chi_s(t)}{\mu(s^*, s]}$$

we obtain

$$(1.3) \qquad P[\,|\,(X_s - X_{s^*}) - E(X_s - X_{s^*})\,| > \epsilon\mu(s^*, s]] \leqq 2\rho^{\mu(s^*,s]}.$$

If $\mu(s^*, s] \to \infty$ as $s \to \infty$, (1.3) yields a bound on the rate at which the increments of the process converge to their expectations.

We will show in §3 that our Theorem is applicable to the Wiener and Poisson processes and will derive the appropriate versions of (1.3). We will also indicate how the Theorem implies a convergence rate theorem for a generalized version of the law of large numbers for independent random variables first given in [3].

2. **Proof of the theorem.** The theorem will be proved in two parts. First, we will establish the existence of $\int f(t)d[X_t - EX_t]$ as the limit-in-the-mean of certain "natural" approximating sums. Inequality 1.2 will be derived in the second part of the proof.

Part 1 *of proof.* We first establish the following lemma.

LEMMA. *Let X be a random variable with $EX = 0$ and with the property that for every $\beta > 0$ there exists $T_\beta > 0$ such that for $|\lambda| \leqq T_\beta$,*

$$(2.1) \qquad Ee^{\lambda x} \leqq e^{\beta R|\lambda|}.$$

Then there exists a number $K_\beta > 0$, depending on β and T_β but not otherwise on the distribution of X, such that

$$\lambda^2 EX^2 \leqq K_\beta R|\lambda|, \quad for \ |\lambda| \leqq T_\beta.$$

PROOF. From (2.1) it follows that for every $\beta > 0$ there exists $\epsilon_\beta > 0$ such that for $|\lambda| \leqq T_\beta$,

$$Ee^{\lambda x} \leqq 1 + (\beta + \epsilon_\beta)|\lambda| R.$$

Then, $E(e^{\lambda x} + e^{-\lambda x}) \leqq 2 + 2(\beta + \epsilon_\beta)|\lambda| R$ for $|\lambda| \leqq T_\beta$ and, since

$$e^{\lambda x} + e^{-\lambda x} = \sum_{k=0}^{\infty} \frac{(\lambda x)^{2k}}{(2k)!} \geqq 2 + \frac{(\lambda x)^2}{2},$$

it follows that

$$\lambda^2 EX^2 \leqq 4(\beta + \epsilon_\beta)R|\lambda|,$$

as was to be shown.

Without loss of generality we will assume $E(X_t - X_s) = 0$ for all $-\infty < s \leqq t < \infty$. Suppose $f \in L_1(\mu) \cap L_\infty(\mu)$ and $\|f\|_1 \leqq 1$. Then there exists a sequence of step functions $f_n=\sum_i c_{ni}\chi_{ni}$ such that $\|f_n-f\|_1 \to 0$, $\|f_n\|_1\leqq 1$ and $\sup_i |c_{ni}|\leqq \|f\|_\infty$ for all n. Here, χ_{ni} is the set characteristic function of $\Delta_{ni}=(a_{n,i-1}, a_{n,i}]$, where $-\infty=a_{n,-\infty}<\cdots<a_{n,0}<a_{n,1}<\cdots<a_{n,\infty}=\infty$, and c_{ni} is the value of f_n on Δ_{ni}. All but a finite number of the c_{ni} can be taken equal to zero for each n.

If $\Delta=(a, b]$, let $X(\Delta)=X_b-X_a$. For each n form the stochastic integral

$$Y_n=\int f_n(t)dX_t=\sum c_{ni}X(\Delta_{ni}).$$

We will show that the sequence $\langle Y_n\rangle$ is a Cauchy sequence in the stochastic Lebesgue space $\mathfrak{L}_2$.

Fix m and n and let Δ_i be the element with index i of the partition obtained by ordering the merged partition $\{\Delta_{mi}\cap\Delta_{nj}: -\infty\leqq i,j\leqq\infty\}$. Let $c_i^{(m)}$ and $c_i^{(n)}$ be the values of f_m and f_n respectively on Δ_i. Then, by the independent increments assumption

$$E(Y_m-Y_n)^2=\sum_i (c_i^{(m)}-c_i^{(n)})^2EX^2(\Delta_i).$$

Let $r=2\|f\|_\infty$, and let

$$\lambda_i=\frac{c_i^{(m)}-c_i^{(n)}}{r}T_\beta \quad \text{for fixed } \beta>0.$$

Then, by Condition 1.1 and the lemma,

$$\begin{aligned} E(Y_m-Y_n)^2&=\frac{r^2}{T_\beta^2}\sum_i \lambda_i^2EX^2(\Delta_i)\\ &\leqq \frac{K_\beta r^2}{T_\beta^2}\sum_i |\lambda_i|\,\mu(\Delta_i)\\ &\leqq \frac{K_\beta r}{T_\beta}\sum_i \left|c_i^{(m)}-c_i^{(n)}\right|\mu(\Delta_i)\\ &\leqq \frac{2K_\beta\|f\|_\infty}{T_\beta}\|f_m-f_n\|_1. \end{aligned}$$

Then, $\langle Y_n\rangle$ is Cauchy in $\mathfrak{L}_2$ which implies the existence of $Y\in\mathfrak{L}_2$ such that $E(Y-Y_n)^2\to 0$. It is easily shown that Y is independent of the sequence $\langle f_n\rangle$ tending to f, so it is proper to write $Y=\int f(t)dX_t$ (see, e.g., [2]).

Part 2 *of proof.* Fix $\epsilon>0$. By a well-known inequality [4, p. 157], for all $\lambda \geqq 0$,

$$P[\pm Y_n > \epsilon] = P\left[\pm \sum_i c_{ni} X(\Delta_{ni}) > \epsilon\right]$$

$$\leqq E \exp\left\{\pm\lambda \sum_i c_{ni} X(\Delta_{ni}) - \lambda\epsilon\right\}$$

$$= e^{-\lambda\epsilon} \prod_i E \exp[\pm\lambda c_{ni} X(\Delta_{ni})].$$

Set $\beta=\epsilon/2$ and $\lambda=T_\beta/\|f\|_\infty$. Then, $|\lambda c_{ni}| \leqq T_\beta$ and from Condition 1.1,

$$P[\pm Y_n > \epsilon] \leqq e^{-\lambda\epsilon} \exp\{\beta|\lambda| \, \|f_n\|_1\}$$
$$\leqq (\exp(-T_\beta\epsilon/2))^{1/\|f\|_\infty}.$$

Inequality (1.2) now follows with $\rho=\exp(-T_\beta\epsilon/2)$ because the $\mathfrak{L}_2$ convergence of $\langle Y_n\rangle$ implies $P[\pm Y_n>\epsilon]\to P[\pm\int f(t)dX_t>\epsilon]$ and because of the inequality $P[|X|>\epsilon]\leqq P[X>\epsilon]+P[-X>\epsilon]$.

3. **Applications of the Theorem.** 1. If X is the Wiener process, X_t: $N(0, \sigma^2 t)$, Condition (1.1) is satisfied with $\mu=\sigma^2\times$ Lebesgue measure and $T_\beta=2\beta$. Then, for $0\leqq s<t<\infty$, Inequality (1.3) becomes

$$P[|X_t - X_s| \geqq \sigma^2(t-s)\epsilon] \leqq 2\rho^{\sigma^2(t-s)}.$$

This is comparable with the inequality given in [2, p. 392].

2. If X is a Poisson process, X_t-X_s: $\mathcal{P}(\mu_{s,t})$, where $\mu_{s,t}=\mu(s, t]$ for a σ-finite measure μ, then Condition (1.1) is satisfied with this measure and T_β the root of largest modulus of the equation $e^\lambda-\lambda-1=\beta|\lambda|$. It follows from Inequality (1.3) that for $-\infty<s<t<\infty$

$$P[|(X_t - X_s) - \mu_{s,t}| > \epsilon\mu_{s,t}] \leqq 2\rho^{\mu_{s,t}}.$$

3. Let μ be counting measure on the integers and let $Y_n=X_n-X_{n-1}$. If $f\in L_1(\mu)\cap L_\infty(\mu)$ is the doubly infinite sequence $\langle a_k\rangle_{k=-\infty}^{\infty}$ then the following is an immediate corollary of our Theorem.

COROLLARY. *Let $\langle Y_n\rangle_{n=-\infty}^{\infty}$ be an independent sequence of random variables such that $EY_n=0$ all n, and for every $\beta>0$ there exists $T_\beta>0$ such that for $|\lambda|\leqq T_\beta$,*

$$E\exp(\lambda Y_n) \leqq \exp(\beta|\lambda|) \quad \textit{uniformly in } n.$$

Let $\langle a_k\rangle_{k=-\infty}^{\infty}$ be a sequence of real constants such that

(i) $\sum_{k=-\infty}^{\infty} |a_k| \leqq 1$,

(ii) $\max_k |a_k| = M < \infty$.

Then, $S = \sum_{k=-\infty}^{\infty} a_k Y_k$ *is well defined as a limit-in-the-mean of order* 2 *of its partial sums and for every* $\epsilon > 0$ *there exists a positive number* $\rho < 1$ *such that*

$$P[|S| > \epsilon] \leqq 2\rho^{1/M}.$$

This theorem generalizes a convergence rate result for the law of large numbers originally established by Cramér [1].

A slightly stronger version of this theorem was proved in [3].

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