A HÖLDER TYPE INEQUALITY FOR SYMMETRIC MATRICES WITH NONNEGATIVE ENTRIES

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The element \( w = (w_1, w_2, \ldots, w_n) \) of the \( n \)-dimensional real euclidean vector space \( \mathbb{R}_n \) is nonnegative if \( 0 \leq w_j \) for each \( j \). If \( 1 \leq k \leq n \) then \( w(k) = (w(k)_1, w(k)_2, \ldots, w(k)_{n-1}) \in \mathbb{R}_{n-1} \) is defined by setting \( w(k)_i = w_i \) if \( 1 \leq i < k \), \( w(k)_i = w_{i+1} \) if \( k \leq i < n \). The real \( n \) by \( n \) matrix \( S = (s_{ij}) \) is nonnegative if \( 0 \leq s_{ij} \) for each \( i, j \). If \( 1 \leq k \leq n \) let \( S(k) \) be the \( n-1 \) by \( n-1 \) matrix obtained by deleting the \( k \)th row and \( k \)th column of \( S \). \( W_n \) is the boundary of the nonnegative cone in \( \mathbb{R}_n \) and \( U_n = \{ u \in \mathbb{R}_n : (u, u) = 1 \} \) is the unit sphere.

Theorem. If \( S \) is a nonnegative symmetric \( n \) by \( n \) matrix, \( u \in U_n \) is nonnegative and \( k \) is a positive integer then \((u, Sw)^k \leq (u, S^ku)\). If \( k > 1 \) equality holds if and only if \( u \) is a characteristic vector of \( S \) or \((u, S^ku) = 0\).

Proof. There is no loss of generality in ignoring trivial cases and assuming that \( k > 1 \), \( n > 1 \), that \( |\lambda| \leq 1 \) for each characteristic value \( \lambda \) of \( S \) and that there is a characteristic value \( \lambda^* \) of \( S \) for which \( |\lambda^*| = 1 \). There is thus a nonnegative characteristic \( n \)-vector \( v \in U_n \) of \( S \) whose corresponding characteristic value \( \lambda \) is \( 1 \) [1, p. 80]. Now proceed by induction on \( n \).

If \( w \in W_n \cap U_n \) there is some \( j \) such that \( w(j) \in U_{n-1} \). If

\[
(w(j), S(j)w(j))^k < (w(j), S^kw(j))
\]

then

\[
(w, Sw)^k = (w(j), S(j)w(j))^k < (w(j), S^kw(j)) \leq (w, S^kw).
\]

If, on the other hand, \( 0 < (w(j), S(j)w(j))^k = (w(j), S^kw(j)) \) then \( w(j) \) is, as a consequence of the induction hypothesis, a characteristic \((n-1)\)-vector of \( S(j) \) and there is some \( \lambda > 0 \) such that \( S(j)w(j) = \lambda w(j) \). Hence \( Sw = \lambda w + p \), where \( p \) is a nonnegative \( n \)-vector for which \((p, w) = 0\). If \( w \) is not a characteristic vector of \( S \) then \((p, p) > 0\) and it is easy to verify, using the symmetry of \( S \), that

\[
(w, S^kw) \geq \lambda^k + \lambda^{k-2}(w, Sp) = \lambda^k + \lambda^{k-2}(p, p) > \lambda^k = (w, Sw)^k.
\]

Thus the truth of the theorem in the \((n-1)\)-dimensional case entails its truth for vectors in \( W_n \).
Suppose the nonnegative vector \( u \in U_n \sim W_n \) is not a characteristic vector of \( S \). Let \( m \in U_n \) be a nonnegative characteristic vector of \( S \) with characteristic value 1 and let \( q \) be the unique element of \( U_n \) orthogonal to \( m \) such that \( u \) is between \( q \) and \( m \) in the sense that there is some \( \eta_0, 0 < \eta_0 < 1 \), for which \( u = (1 - \eta_0)^{1/2}m + \eta_0q \). Let \( \alpha = (q, S^kq) - 1, \beta = (q, Sq) - 1 \). Notice that \( \beta < 0 \), for otherwise it would follow from the normalization of \( S \) that \( q \) would be a characteristic vector of \( S \) with characteristic value 1, whence so would \( u \), contrary to assumption. There is some \( w \in W_n \cap U_n \) which lies between \( u \) and \( q \), that is there is some \( \eta_1, 0 < \eta_1 \leq 1 \), such that \( (1 - \eta_1)^{1/2}m + \eta_1q = w \).

Let \( f(\lambda) = \lambda^k - \lambda\alpha/\beta - 1 + \alpha/\beta \) for each real \( \lambda \). Then

\[
\begin{align*}
 f(1) &= (m, Sm)^k - (m, S^k m) = 0, \\
 f(1 + \eta_0\beta) &= (u, Su)^k - (u, S^k u), \quad \text{and} \\
 f(1 + \eta_1\beta) &= (w, Sw)^k - (w, S^k w) \leq 0
\end{align*}
\]

as a consequence of the symmetry of \( S \). Since \( 0 < 1 + \eta_0^2\beta < 1 + \eta_1^2\beta < 1 \) and \( f \) is a strictly convex [2, p. 75] function of a positive argument strict inequality holds at \( u \).

References


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