A HÖLDER TYPE INEQUALITY FOR SYMMETRIC MATRICES WITH NONNEGATIVE ENTRIES

G. R. BLAKLEY AND PRABIR ROY

The element $w=(w_1, w_2, \cdots, w_n)$ of the n-dimensional real euclidean vector space R_n is nonnegative if $0 \leqq w_j$ for each j. If $1 \leqq k \leqq n$ then $w(k)=(w(k)_1, w(k)_2, \cdots, w(k)_{n-1}) \in R_{n-1}$ is defined by setting $w(k)_i = w_i$ if $1 \leqq i < k$, $w(k)_i = w_{i+1}$ if $k \leqq i < n$. The real n by n matrix $S=(s_{ij})$ is nonnegative if $0 \leqq s_{ij}$ for each i, j. If $1 \leqq k \leqq n$ let $S(k)$ be the $n-1$ by $n-1$ matrix obtained by deleting the kth row and kth column of S. W_n is the boundary of the nonnegative cone in R_n and $U_n = \{u \in R_n: (u, u)=1\}$ is the unit sphere.

THEOREM. *If S is a nonnegative symmetric n by n matrix, $u \in U_n$ is nonnegative and k is a positive integer then $(u, Su)^k \leqq (u, S^k u)$. If $k>1$ equality holds if and only if u is a characteristic vector of S or $(u, S^k u)$ $=0$.*

PROOF. There is no loss of generality in ignoring trivial cases and assuming that $k>1$, $n>1$, that $|\lambda| \leqq 1$ for each characteristic value λ of S and that there is a characteristic value λ^* of S for which $|\lambda^*|=1$. There is thus a nonnegative characteristic n-vector $v \in U_n$ of S whose corresponding characteristic value λ is 1 [1, p. 80]. Now proceed by induction on n.

If $w \in W_n \cap U_n$ there is some j such that $w(j) \in U_{n-1}$. If

$$(w(j), S(j)w(j))^k < (w(j), S(j)^k w(j))$$

then

$$(w, Sw)^k = (w(j), S(j)w(j))^k < (w(j), S(j)^k w(j)) \leqq (w, S^k w).$$

If, on the other hand, $0<(w(j), S(j)w(j))^k = (w(j), S(j)^k w(j))$ then $w(j)$ is, as a consequence of the induction hypothesis, a characteristic $(n-1)$-vector of $S(j)$ and there is some $\lambda>0$ such that $S(j)w(j)$ $=\lambda w(j)$. Hence $Sw=\lambda w+p$, where p is a nonnegative n-vector for which $(p, w)=0$. If w is not a characteristic vector of S then $(p,p)>0$ and it is easy to verify, using the symmetry of S, that

$$(w, S^k w) \geqq \lambda^k + \lambda^{k-2}(w, Sp) = \lambda^k + \lambda^{k-2}(p, p) > \lambda^k = (w, Sw)^k.$$

Thus the truth of the theorem in the $(n-1)$-dimensional case entails its truth for vectors in W_n.

Received by the editors November 2, 1963 and, in revised form, October 5, 1964.

Suppose the nonnegative vector $u \in U_n \sim W_n$ is not a characteristic vector of S. Let $m \in U_n$ be a nonnegative characteristic vector of S with characteristic value 1 and let q be the unique element of U_n orthogonal to m such that u is between q and m in the sense that there is some η_0, $0<\eta_0<1$, for which $u=(1-\eta_0^2)^{1/2}m+\eta_0 q$. Let $\alpha=(q, S^k q)-1$, $\beta=(q, Sq)-1$. Notice that $\beta<0$, for otherwise it would follow from the normalization of S that q would be a characteristic vector of S with characteristic value 1, whence so would u, contrary to assumption. There is some $w \in W_n \cap U_n$ which lies between u and q, that is there is some η_1, $\eta_0<\eta_1 \leqq 1$, such that $(1-\eta_1^2)^{1/2}m+\eta_1 q=w$.

Let $f(\lambda)=\lambda^k-\lambda\alpha/\beta-1+\alpha/\beta$ for each real λ. Then

$$f(1) = (m, Sm)^k - (m, S^k m) = 0,$$

$$f(1 + \eta_0^2\beta) = (u, Su)^k - (u, S^k u), \quad \text{and}$$

$$f(1 + \eta_1^2\beta) = (w, Sw)^k - (w, S^k w) \leqq 0$$

as a consequence of the symmetry of S. Since $0<1+\eta_1^2\beta<1+\eta_0^2\beta<1$ and f is a strictly convex [2, p. 75] function of a positive argument strict inequality holds at u.

References

1. F. R. Gantmacher, *Applications of the theory of matrices*, Interscience, New York, 1959.

2. G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1952.

University of Illinois and
University of Wisconsin