A HÖLDER TYPE INEQUALITY FOR SYMMETRIC MATRICES WITH NONNEGATIVE ENTRIES

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The element \( w = (w_1, w_2, \cdots, w_n) \) of the \( n \)-dimensional real euclidean vector space \( R^n \) is nonnegative if \( 0 \leq w_j \) for each \( j \). If \( 1 \leq k \leq n \) then \( w(k) = (w(k)_1, w(k)_2, \cdots, w(k)_{n-1}) \in R^{n-1} \) is defined by setting \( w(k)_i = w_i \) if \( 1 \leq i < k \), \( w(k)_i = w_{i+1} \) if \( k \leq i < n \). The real \( n \) by \( n \) matrix \( S = (s_{ij}) \) is nonnegative if \( 0 \leq s_{ij} \) for each \( i, j \). If \( 1 \leq k \leq n \) let \( S(k) \) be the \( n-1 \) by \( n-1 \) matrix obtained by deleting the \( k \)th row and \( k \)th column of \( S \). \( W_n \) is the boundary of the nonnegative cone in \( R_n \) and \( U_n = \{ u \in R_n : (u, u) = 1 \} \) is the unit sphere.

Theorem. If \( S \) is a nonnegative symmetric \( n \) by \( n \) matrix, \( u \in U_n \) is nonnegative and \( k \) is a positive integer then \( (u, Su)^k \leq (u, S^k u) \). If \( k > 1 \) equality holds if and only if \( u \) is a characteristic vector of \( S \) or \( (u, S^k u) = 0 \).

Proof. There is no loss of generality in ignoring trivial cases and assuming that \( k > 1 \), \( n > 1 \), that \( |\lambda| \leq 1 \) for each characteristic value \( \lambda \) of \( S \) and that there is a characteristic value \( \lambda^* \) of \( S \) for which \( |\lambda^*| = 1 \). There is thus a nonnegative characteristic \( n \)-vector \( v \in U_n \) of \( S \) whose corresponding characteristic value \( \lambda \) is 1 [1, p. 80]. Now proceed by induction on \( n \).

If \( w \in W_n \cap U_n \) there is some \( j \) such that \( w(j) \in U_{n-1} \). If

\[
(w(j), S(j)w(j))^k < (w(j), S^k w(j))
\]

then

\[
(w, Sw)^k = (w(j), S(j)w(j))^k < (w(j), S^k w(j)) \leq (w, S^k w).
\]

If, on the other hand, \( 0 < (w(j), S(j)w(j))^k = (w(j), S^k w(j)) \) then \( w(j) \) is, as a consequence of the induction hypothesis, a characteristic \((n-1)\)-vector of \( S(j) \) and there is some \( \lambda > 0 \) such that \( S(j)w(j) = \lambda w(j) \). Hence \( Sw = \lambda w + p \), where \( p \) is a nonnegative \( n \)-vector for which \( (p, w) = 0 \). If \( w \) is not a characteristic vector of \( S \) then \( (p, p) > 0 \) and it is easy to verify, using the symmetry of \( S \), that

\[
(w, S^k w) \geq (w, Sw)^k = (w, S^k w) + (w, S^{k-1} p) = \lambda^k + \lambda^{k-1}(p, p) > \lambda^k = (w, Sw)^k.
\]

Thus the truth of the theorem in the \((n-1)\)-dimensional case entails its truth for vectors in \( W_n \).

Received by the editors November 2, 1963 and, in revised form, October 5, 1964.

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Suppose the nonnegative vector \( u \in U_n \) is not a characteristic vector of \( S \). Let \( m \in U_n \) be a nonnegative characteristic vector of \( S \) with characteristic value 1 and let \( q \) be the unique element of \( U_n \) orthogonal to \( m \) such that \( u \) is between \( q \) and \( m \) in the sense that there is some \( \eta_0, 0 < \eta_0 < 1 \), for which \( u = (1 - \eta_0)^{1/2} m + \eta_0 q \). Let \( \alpha = (q, S^k q) - 1, \beta = (q, S q) - 1 \). Notice that \( \beta < 0 \), for otherwise it would follow from the normalization of \( S \) that \( q \) would be a characteristic vector of \( S \) with characteristic value 1, whence so would \( u \), contrary to assumption. There is some \( w \in W_n \cap U_n \) which lies between \( u \) and \( q \), that is there is some \( \gamma_1, \gamma_0 < \gamma_1 < 1 \), such that \( (1 - \gamma_0)^{1/2} m + \gamma_1 q = w \).

Let \( f(\lambda) = \lambda^k - \lambda \alpha / \beta - 1 + \alpha / \beta \) for each real \( \lambda \). Then

\[
f(1) = (m, S^k m) - (m, S^k m) = 0,
\]

\[
f(1 + \eta_0 \beta) = (u, S^k u) - (u, S^k u), \quad \text{and}
\]

\[
f(1 + \eta_1 \beta) = (w, S w) - (w, S^k w) \leq 0
\]
as a consequence of the symmetry of \( S \). Since \( 0 < 1 + \eta_0 \beta < 1 + \eta_1 \beta < 1 \) and \( f \) is a strictly convex \([2, \text{p. 75}]\) function of a positive argument strict inequality holds at \( u \).

**References**


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