

LIE GROUPS AND PRODUCTS OF SPHERES¹

P. G. KUMPEL, JR.

1. Introduction. In this note, we discuss a problem which was suggested by Serre [12]. Throughout the paper, G denotes a compact, connected, simply connected, simple Lie group. It is a well-known result of Hopf that the real cohomology of G is isomorphic to that of a space $X(G)$, which is a product of odd dimensional spheres:

$$X(G) = S^{n_1} \times \cdots \times S^{n_l}, \quad l = \text{rank } G, \quad \sum_{i=1}^l n_i = \dim G = n.$$

Serre defines a prime p to be *regular* for G if there exists a map $f: X(G) \rightarrow G$ such that $f_*: H_i(X(G); Z_p) \rightarrow H_i(G; Z_p)$ is an isomorphism for $i \geq 0$. In the same paper [12], it is shown that

THEOREM 1.1 (SERRE). *If $p \geq (\dim G/\text{rank } G) - 1$, then p is regular for G . If, in addition, G is a classical group, the inequality is necessary for the regularity of p .*

We prove here the necessity of the inequality $p \geq (\dim G/\text{rank } G) - 1$ for regularity of p for exceptional Lie groups G and thereby obtain

THEOREM 1.2. *A prime p is regular for G if and only if $p \geq (\dim G/\text{rank } G) - 1$.*

The proof of 1.2, which uses the classification of compact, simple Lie groups, depends heavily on a method of Clark [6] for finding nonzero Steenrod reduced powers.

In his paper [3], Borel calls attention to Serre's work on the problem of comparing G and $X(G)$. He, in fact, states that the above inequality is *not* necessary in the case of G exceptional because of a result of Toda stating that $\pi_{10}(G_2) = Z_3$. This error was later corrected by Bott and Samelson [5]. In §3, we give a simple proof of the Bott-Samelson result that $\pi_{10}(G_2) = 0$ using some calculations of Kervaire [10] and a result stated by Harris [7], the proof of which is contained in a forthcoming paper of this author [11].

The author is indebted to Professor Allan Clark for calling to his attention the methods used here, and to Professor Bruno Harris for his advice during the preparation of the paper. This result is con-

Received by the editors November 6, 1964.

¹ This research was supported in part by NSF Grant 24026.

tained in the author's doctoral dissertation and was presented to the American Mathematical Society in February 1964 (Abstract 609-7, Notices Amer. Math. Soc. 11 (1964), 205-207).

2. **Proof of Theorem 1.2.** In view of Serre's result (1.1), it suffices to show that for exceptional G , if $p < (\dim G/\text{rank } G) - 1$, then p is irregular for G .

Since spheres have no torsion, if G has p -torsion, then p is irregular for G . Using the definition of regularity, together with the J. H. C. Whitehead theorem (see [12, p. 276]), we find that if p is regular for G , then the p -primary component of $\pi_j(G)$ is isomorphic with the direct sum of the p -primary components of $\pi_j(S^{n_i})$, $i = 1, \dots, l$. Also, using the naturality of the reduced powers of Steenrod, if there is a nonzero power $P_p^i: H^i(G; Z_p) \rightarrow H^{i+2i(p-1)}(G; Z_p)$ then p is irregular for G ; for the reduced powers are trivial for products of spheres. Hence we may state the

LEMMA 2.1. *Each of the following implies that p is irregular for G : (a) G has p -torsion, (b) $\pi_j(G)$ and $\bigoplus_{i=1}^j \pi_j(S^{n_i})$ do not have isomorphic p -primary components, (c) there is a nonzero $P_p^i: H^i(G; Z_p) \rightarrow H^{i+2i(p-1)}(G; Z_p)$ for some $i \geq 1$.*

In table (2.2), we record for each exceptional group G , those primes p for which G has p -torsion (see Borel [2]), together with the numbers $\dim G$, $(\dim G/\text{rank } G) - 1$, and the sequences (n_1, \dots, n_l) of dimensions of spheres in $X(G)$ (see Borel and Chevalley [4]).

We proceed to list, for each exceptional group G , those primes $p < (\dim G/\text{rank } G) - 1$, such that G has no p -torsion, $p \neq 3$ if $G = G_2$. For such p we note the pairs (m_j, m_k) satisfying the hypotheses of Clark's theorem (2.3). We then record the corresponding nonzero P_p^1 .

(2.2)

G	$\dim G$	$\left(\frac{\dim G}{\text{rank } G}\right) - 1$	p -torsion	(n_1, \dots, n_l)
G_2	14	6	2	(3, 11)
F_4	52	12	2, 3	(3, 11, 15, 23)
E_6	78	12	2, 3	(3, 9, 11, 15, 17, 23)
E_7	133	18	2, 3	(3, 11, 15, 19, 23, 27, 35)
E_8	248	30	2, 3, 5	(3, 15, 23, 27, 35, 39, 47, 59)

Bott and Samelson [5] have shown that $\pi_{10}(G_2)$ has 3-primary

component zero. Since $\pi_{10}(S^3 \times S^{11})_3 = Z_3$ (see [12, p. 285]), by (2.1b), 3 is irregular for G_2 . (See also Proposition 3.1.) From the table (2.2) of p -torsion, and (2.1a), together with the preceding remark, it follows that 2 and 3 are irregular for all exceptional groups, and that 5 is irregular for E_8 .

Now we show that for exceptional G , if $p < (\dim G/\text{rank } G) - 1$, $p \neq 2, 3$ and $p \neq 5$ if $G = E_8$, there exists a nonzero reduced power $P_p^i: H^i(G; Z_p) \rightarrow H^{i+2i(p-1)}(G; Z_p)$. This result will complete the proof of (1.2) using (2.1c).

The following is a rewording of a result due to Clark [6]. We state and prove it in a form which is convenient for our purposes.

THEOREM 2.3 (CLARK). *Let G be as in (1.2). Let $H^*(G; R)$ be an exterior algebra on generators $x_{n_i}, i = 1, \dots, l, \deg x_{n_i} = n_i = 2m_i - 1$. If p is prime and G has no p -torsion and if there exists $k, 1 \leq k \leq l$, such that (i) $m_k \not\equiv 0 \pmod{p}$, (ii) $m_k > p$, (iii) the set $\{m_1, \dots, m_l\}$ contains exactly one element m_j such that $m_j \equiv (1-p) \pmod{m_k}$, and $m_j < m_k$, then $P_p^1 x_{2m_j-1} = x_{2m_k-1}$.*

PROOF OF (2.3). Since G has no p -torsion, the classifying space B of G has the property that $H^*(B; Z_p)$ is a polynomial algebra on generators $y_{2m_i}, i = 1, \dots, l$, which suspend to the generators x_{2m_i-1} . Let m_k be as in the statement of the theorem. Let I be the ideal of $H^*(B; Z_p)$ generated by the $y_{2m_i}, i \neq k$. Then either $P_p^1(I) \subset I$ or $P_p^1(I) \not\subset I$.

We shall now see that the first of these is impossible. Since $m_k > p$, we may write $m_k = a + bp, b > 0, 0 < a < p$. As a special case of the Adem relations, [1], $P_p^a P_p^{bp} y_{2m_k} = P_p^{m_k} y_{2m_k} = y_{2m_k}^p$. Since $P_p^1(I) \subset I$, we have $P_p^a(I) \subset I$, for $a! P_p^a = (P_p^1)^a$, and $a! \not\equiv 0 \pmod{p}$. Hence, if $P_p^{bp} y_{2m_k} \in I$, we have $y_{2m_k}^p \in I$, which is a contradiction. Therefore $P_p^{bp} y_{2m_k} \notin I$. Thus, $P_p^{bp} y_{2m_k} = qy_{2m_k}^r + s, q \neq 0, q \in Z_p, s \in I$. Taking degrees we get $2m_k + 2bp(p-1) = 2rm_k$. Hence $m_k(r-1) = bp(p-1)$. Since $m_k > bp, r < p$, and since p is prime, and $r-1$ divides $bp(p-1)$, we see that $r-1$ divides $b(p-1)$. Hence $m_k \equiv 0 \pmod{p}$, a contradiction. Thus $P_p^1(I) \not\subset I$.

If all generators of I were mapped into I by P_p^1 , we would have $P_p^1(I) \subset I$. For an element of I can be written as a sum of elements of the form $u_i y_{2m_i}, u_i \in H^*(B; Z_p)$. But P_p^1 preserves sums and $P_p^1(u_i y_{2m_i}) = P_p^1(u_i) y_{2m_i} + \epsilon u_i P_p^1 y_{2m_i}, \epsilon = \pm 1$, which is in I if $P_p^1(y_{2m_i}) \in I$. Therefore $P_p^1(y_{2m_i}) \in I$ for some $i \neq k$, that is $P_p^1 y_{2m_i} = qy_{2m_i}^r + s, q \neq 0, s \in I$.

Taking degrees, we get $2m_i + 2(p-1) = (r)(2m_i)$, i.e., $m_i = (1-p) + rm_k$. Hence $m_i \equiv (1-p) \pmod{m_k}$. By hypothesis, there is only one

such m_i , called m_j . Moreover, $m_j < m_k$. We show that $r = 1$. If $r > 1$, then $(r - 1)m_k > p$, hence $(r - 1)m_k + m_k - m_j > p$, or $rm_k - m_j > p > p - 1$, i.e., $m_j < (1 - p) + rm_k$ contrary to the fact that $m_j = (1 - p) + rm_k$. Hence $m_j = (1 - p) + m_k$, and $P_p^1 y_{2m_j} = qy_{2m_k} + s$. Since s is decomposable, it is mapped into zero by the suspension, and by the naturality we obtain $P_p^1 x_{2m_j-1} = x_{2m_k-1}$. We may assume $q = 1$ by suitably choosing the generators x_i . This proves (2.3).

We now apply (2.3) to complete the proof of (1.2). First note from (2.2) that for G_2, F_4, E_6, E_7, E_8 , the sequences $(m_1, \dots, m_i), m_i = (n_i + 1)/2$ are, respectively, (2, 6), (2, 6, 8, 12), (2, 5, 6, 8, 9, 12), (2, 6, 8, 10, 12, 14, 18), and (2, 8, 12, 14, 18, 20, 24, 30).

G	p	(m_j, m_k)	nonzero P_p^1
G_2	5	(2, 6)	$P_5^1 x_3 = x_{11}$
F_4	5	(8, 12)	$P_5^1 x_{15} = x_{23}$
	7	(2, 8)	$P_7^1 x_3 = x_{15}$
		(6, 12)	$P_7^1 x_{11} = x_{23}$
	11	(2, 12)	$P_{11}^1 x_3 = x_{23}$
E_6	5	(5, 9)	$P_5^1 x_9 = x_{17}$
		(8, 12)	$P_5^1 x_{15} = x_{23}$
	7	(2, 8)	$P_7^1 x_3 = x_{15}$
		(6, 12)	$P_7^1 x_{11} = x_{23}$
	11	(2, 12)	$P_{11}^1 x_3 = x_{23}$
E_7	5	(8, 12)	$P_5^1 x_{15} = x_{23}$
		(10, 14)	$P_5^1 x_{19} = x_{27}$
		(14, 18)	$P_5^1 x_{27} = x_{35}$
	7	(12, 18)	$P_7^1 x_{23} = x_{35}$
	11	(8, 18)	$P_{11}^1 x_{15} = x_{35}$
	13	(2, 14)	$P_{13}^1 x_3 = x_{27}$
		(6, 18)	$P_{13}^1 x_{11} = x_{35}$
	17	(2, 18)	$P_{17}^1 x_3 = x_{35}$

G	p	(m_j, m_k)	nonzero P_p^1
E_8	7	(14, 20)	$P_7^1 x_{27} = x_{39}$
		(18, 24)	$P_7^1 x_{35} = x_{47}$
		(24, 30)	$P_7^1 x_{47} = x_{59}$
	11	(8, 18)	$P_{11}^1 x_{15} = x_{35}$
		(14, 24)	$P_{11}^1 x_{27} = x_{47}$
		(20, 30)	$P_{11}^1 x_{39} = x_{59}$
	13	(8, 20)	$P_{13}^1 x_{15} = x_{39}$
		(12, 24)	$P_{13}^1 x_{23} = x_{47}$
		(18, 30)	$P_{13}^1 x_{35} = x_{59}$
	17	(8, 24)	$P_{17}^1 x_{15} = x_{47}$
		(14, 30)	$P_{17}^1 x_{27} = x_{59}$
	19	(2, 20)	$P_{19}^1 x_3 = x_{39}$
		(12, 30)	$P_{19}^1 x_{23} = x_{59}$
	23	(2, 24)	$P_{23}^1 x_3 = x_{47}$
		(8, 30)	$P_{23}^1 x_{15} = x_{59}$
29	(2, 30)	$P_{29}^1 x_3 = x_{59}$	

In each case, the fact that (m_j, m_k) satisfies the hypotheses of Theorem 2.3 is easily checked. Thus (1.2) is established.

3. **Computation of $\pi_{10}(G_2)$.** Finally we prove the following proposition, whose significance was noted in the introduction.

PROPOSITION 3.1. $\pi_{10}(G_2) = 0$.

PROOF. In Harris's paper [7], it is stated that the homotopy exact sequence of the fibration $G_2 \rightarrow \text{Spin } 8 \rightarrow \text{Spin}/G_2$ is split when tensored with Q_3 , the ring of rational numbers whose denominators are powers of 3. The splitting is given by the map

$$q: \text{Spin } 8/G_2 \rightarrow \text{Spin } 8, q(\tau G_2) = \tau\sigma(\tau)^{-1}, \tau \in \text{Spin } 8,$$

σ the automorphism of order 3 of Spin 8 having G_2 as fixed point set.

(A description of the inclusion $G_2 \subset \text{Spin } 8$ and a proof of this splitting are contained in a forthcoming paper of the author [11].)

Let $p: \text{Spin } 8 \rightarrow \text{Spin } 8/G_2$ be the projection. It is known that $\text{Spin } 8/G_2$ is homeomorphic with $S^7 \times S^7$ (see [9, II, p. 93]). In the proof of the above splitting, it was shown that if we identify

$$\pi_j(\text{Spin } 8/G_2) \approx \pi_j(S^7) \oplus \pi_j(S^7),$$

then

$$p_*q_*(\alpha, \beta) = (\alpha + \beta, 2\beta - \alpha), \quad \alpha, \beta \in \pi_j(S^7).$$

We record the following results on homotopy groups of spheres and orthogonal groups:

$$\begin{aligned} \pi_{n+4}(S^n) &= 0, & n &\geq 6, \\ \pi_{n+3}(S^n) &= Z_{24}, & n &\geq 5, \\ \pi_{8s+2}(SO(8s)) &= Z_{24} \oplus Z_8, & s &\geq 1. \end{aligned}$$

The first two can be found in Hu [8, pp. 329–331], and the last is due to Kervaire [10].

Take $n=7, s=1$, and observe that the exact homotopy sequence of the above fibering

$$\dots \rightarrow \pi_{11}(S^7 \times S^7) \xrightarrow{\partial} \pi_{10}(G_2) \xrightarrow{j^*} \pi_{10}(\text{Spin } 8) \xrightarrow{p^*} \pi_{10}(S^7 \times S^7) \rightarrow \dots$$

reduces to the exact sequence

$$0 \rightarrow \pi_{10}(G_2) \xrightarrow{j^*} Z_{24} \oplus Z_8 \xrightarrow{p^*} Z_{24} \oplus Z_{24}$$

since $\pi_n(\text{Spin } 8) \approx \pi_n(SO(8)), n \geq 2$.

If $\pi_{10}(G_2)$ has an element x of order 3, then $j_*(x) \neq 0$ and has order 3, therefore generates the 3-component of $\pi_{10}(\text{Spin } 8) \approx Z_{24} \oplus Z_8 \approx Z_3 \oplus Z_8 \oplus Z_8$. Since $p_*j_* = 0$, all elements of order 3 are in $\ker p_*$. Let $\alpha \in \pi_{10}(S^7) \approx Z_{24}$ be an element of order 3, $\alpha \neq 0$. Then $q_*(\alpha, 0) \in \pi_{10}(\text{Spin } 8)$ has order 3 or is zero. In any case, $0 = p_*q_*(\alpha, 0) = (\alpha, -\alpha)$, hence $\alpha = 0$, a contradiction. Hence $\pi_{10}(G_2)_3 = 0$.

Finally, from Serre [12, Corollary 2, p. 289], we know that $\pi_{10}(G_2)$ is finite. From the fact that the homotopy sequence of $G_2 \rightarrow \text{Spin } 8 \rightarrow \text{Spin } 8/G_2$ is split upon tensoring with Q_3 , we obtain

$$(\pi_{10}(G_2) \oplus \pi_{10}(S^7 \times S^7)) \otimes Q_3 \approx \pi_{10}(\text{Spin } 8) \otimes Q_3.$$

Therefore

$$(\pi_{10}(G_2) \otimes Q_3) \oplus Z_8 \oplus Z_8 \approx Z_8 \oplus Z_8.$$

Hence $\pi_{10}(G_2) \otimes Q_3 = 0$, i.e., $\pi_{10}(G_2)$ has no elements of finite order $\neq 3$. We have seen that $\pi_{10}(G_2)_3 = 0$, hence 3.1 is proved.

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