AN L^1 EXTREMAL PROBLEM FOR POLYNOMIALS

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There are a large number of polynomial extremal problems all of which seem to ask the same question. In very crude terms it is the following one: How closely can we approximate a nontrivial situation where

$$|a_k| \text{ is a constant for } k = 0, 1, 2, \cdots, n$$

and

$$\left|\sum_{k=0}^{n} a_k z^k\right| \text{ is a constant for } |z| = 1?$$

One such extremal problem which has received considerable attention is the following one:

To choose a_k with $|a_k| = 1$ so that $M = \max_{|z|=1} |\sum_{k=0}^{n} a_k z^k|$ is a minimum.

Since

$$M^2 \geqq \frac{1}{2\pi}\int \left|\sum_{k=0}^{n} a_k z^k\right|^2 |dz| = \sum_{k=0}^{n} |a_k|^2 = n + 1$$

it follows that $M \geqq (n+1)^{1/2}$. However, in order that $(n+1)^{1/2}$ be (close to) the right answer the inequality $M^2 \geqq (1/2\pi)\int |\sum a_k z^k|^2 |dz|$ must be (close to) equality, which is to say, $|\sum a_k z^k|$ must be close to constant.

It was proved by Hardy that this minimum M, M_n, satisfies $M_n \leqq c(n+1)^{1/2}$, c some absolute constant (see Zygmund [4]). Shapiro has even shown that the a_k can be chosen real (i.e., equal to ± 1) and the same estimate achieved (see Rudin [3]).

Thus the order of magnitude of M_n is determined as $\sqrt{n}$. The deeper question regarding the limit of $M_n/\sqrt{n}$ remains unsettled. In terms of our original heuristic formulation it makes a vital difference whether $M_n/\sqrt{n} \to 1$ or not. Some partial results in this direction have been obtained by Erdös and Littlewood [1].

Another extremal problem in the same spirit is the following:

Among all polynomials for which

$$\left|\sum_{k=0}^{n} a_k z^k\right| \leqq 1 \quad \text{for } |z| \leqq 1.$$

To find $\max_{\{a_k\}} \sum_{k=0}^{n} |a_k| = \mathfrak{M}_n$.

Here the Schwarz inequality gives

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$$\sum |a_k| \leqq ((n+1)\sum |a_k|^2)^{1/2} = \left(\frac{n+1}{2\pi}\int_{|z|=1}\left|\sum a_k z^k\right|^2 |dz|\right)^{1/2}$$
$$\leqq (n+1)^{1/2}$$

so that $\mathfrak{M}_n \leqq (n+1)^{1/2}$. Again for this estimate to be accurate we must have near equality in all the above estimates. Thus $|a_k|$ should be nearly constant, as should $|\sum a_k z^k|$.

It was shown by Newman that, for $n>0$, $\mathfrak{M}_n \leqq \sqrt{n}$, equality holding for $n=1$, 2, and 4. Shapiro, however, showed that $n=1, 2, 4$ are the only cases of equality. Hardy's example shows that $\mathfrak{M}_n \geqq c\sqrt{n}$, $c>0$, and so settles the order of magnitude of $\mathfrak{M}_n$. Again the deeper problem of whether $\mathfrak{M}_n/\sqrt{n} \to 1$ remains unsettled.

As our third example we consider the problem of maximizing

$$\frac{1}{2\pi}\int\left|\sum_{k=0}^{n} a_k z^k\right| |dz| \text{ subject to } |a_k| = 1 \quad \text{for } k = 0, 1, \cdots, n.$$

If this maximum is called I_n, we obtain $I_n \leqq (n+1)^{1/2}$ by the Schwarz inequality. In the case of real a_i this has been improved to $I_n \leqq (n+.97)^{1/2}$ (for $n>0$) (see Newman [2]). Once more, by the previously cited examples, it follows that $I_n > c\sqrt{n}$, $c>0$, and so the order of magnitude of I_n is again $\sqrt{n}$.

The purpose of the present paper is to prove that $I_n/\sqrt{n} \to 1$, so that, at least in this sense, the meta-problem posed in our introduction is solved. We actually prove somewhat more, namely

THEOREM 1. *$I_n \geqq \sqrt{n} - c$, c an absolute constant.*

The example we have constructed is related to the Gaussian sums and is motivated by the fact that these sums have size $\sqrt{n}$ while consisting of n terms of modulus 1. We set $\omega = \exp(\pi i/(n+1))$, $a_k = \omega^{k^2}$, $k=0, 1, \cdots, m$, $P(z) = \sum_{k=0}^{n} a_k z^k$, and we prove

LEMMA. $(1/2\pi)\int_{|z|=1} |P(z)|^4 |dz| = n^2 + O(n^{3/2})$.

PROOF. Writing $|\sum_{k=0}^{n} a_k e^{ik\theta}|^2 = \sum_{-n}^{n} c_j e^{ij\theta}$, we note that $(1/2\pi)\int_{|z|=1}|P(z)|^4 |dz| = \sum_{-n}^{n} |c_j|^2$. We also see that $c_0 = n+1$. Next we examine c_j for $j>0$. We have

$$c_j = \sum_{k=0}^{n-j} \omega^{(k+j)^2-k^2} = \omega^{j^2}\sum_{k=0}^{n-j}\omega^{2kj} = \omega^{j^2}\frac{1-\omega^{2(n+1-j)j}}{1-\omega^{2j}} = -\omega^{-j}\frac{\omega^{j^2}-\omega^{-j^2}}{\omega^{j}-\omega^{-j}}$$

and we obtain the formula

$$(1) \qquad |c_j|^2 = \frac{\sin^2 j^2 \dfrac{\pi}{n+1}}{\sin^2 j \dfrac{\pi}{n+1}} \qquad \text{for } j = 1, 2, \cdots, n.$$

We proceed to estimate $\sum_{j=1}^{n} |c_j|^2$ by splitting this sum into four parts

$$S_1 = \sum_{j \le \sqrt{n}} |c_j|^2, \qquad S_2 = \sum_{\sqrt{n} < j \le (n+1)/2} |c_j|^2,$$
$$S_3 = \sum_{(n+1)/2 < j < n+1-\sqrt{n}} |c_j|^2, \qquad S_4 = \sum_{n+1-\sqrt{n} \le j \le n} |c_j|^2.$$

First of all we note that, for $j > 0$ by (1)

$$|c_{n+1-j}|^2 = |c_j|^2$$

so that

$$(2) \qquad |S_3| \le |S_2| \quad \text{and} \quad |S_4| \le |S_1|.$$

Next, from the inequality $|\sin j\theta/\sin\theta| \le j$, we conclude that

$$(3) \qquad |S_1| \le \sum_{j \le \sqrt{n}} j^2 \le n \sum_{j \le \sqrt{n}} 1 \le n^{3/2}.$$

Furthermore, from the inequality $|1/\sin\theta| \le \pi/2\theta$ (valid in $0 \le \theta \le \pi/2$) we conclude that

$$(4) \quad |S_2| \le \sum_{\sqrt{n} < j \le (n+1)/2} \frac{(n+1)^2}{4j^2} < \frac{(n+1)^2}{4} \sum_{\sqrt{n} < j < \infty} \frac{1}{j^2} \le \frac{n^2}{\sqrt{n}} = n^{3/2}.$$

If we observe that $c_{-j} = \bar{c}_j$, and combine (2), (3), and (4), we finally obtain

$$(5) \qquad \sum_{j \ne 0} |c_j|^2 \le 8n^{3/2}$$

and the lemma is proved.

PROOF OF THEOREM 1. We use the Hölder inequality to conclude that

$$\int |f|^2 \le \left(\int |f|^4\right)^{1/3} \left(\int |f|\right)^{2/3}$$

which, applied to our present case, gives

$$\frac{1}{2\pi}\int_{|z|=1} |P| \geqq \frac{\left(\frac{1}{2\pi}\int |P|^2\right)^{3/2}}{\left(\frac{1}{2\pi}\int |P|^4\right)^{1/2}} = \frac{(n+1)^{3/2}}{\left(\frac{1}{2\pi}\int |P|^4\right)^{1/2}}$$

so that, applying our lemma, we obtain

$$\frac{1}{2\pi}\int \left|\sum a_k z^k\right| \geqq \frac{(n+1)^{3/2}}{(n^2+An^{3/2})^{1/2}} > \frac{n^{1/2}}{(1+An^{-1/2})^{1/2}}$$
$$> n^{1/2}(1-cn^{-1/2}) = \sqrt{n}-c. \qquad \text{Q.E.D.}$$

Theorem 1 is a statement regarding L^1 norms. By applying the principle of duality we can obtain from it a theorem on L^∞ functions. Because of its independent interest we record the statement of this theorem.

THEOREM 2. *Given $n \geqq 0$, there exists a measurable function, $f(\theta)$, with period 2π and satisfying $|f(\theta)| \leqq 1$ for all θ, such that, with $\{b_k\}$ the Fourier coefficients of $f(\theta)$, we have $\sum_{k=0}^{n} |b_k| \geqq \sqrt{n}-c$, c an absolute constant.*

One further remark is perhaps pertinent. We point out, namely, that the behavior exhibited by our polynomial, $P(z)$, is quite exceptional. Indeed, the crucial fact about $P(z)$ is that its L^4 norm is close to $\sqrt{n}$, and it follows from the work of Paley (see Zygmund [4]) that, in a certain sense, *most* nth degree polynomials with coefficients of modulus 1 have L^4 norms which, instead, are close to $2^{1/4}\sqrt{n}$.

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