Reference

1. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

Louisiana State University

LOWER BOUNDS FOR SOLUTIONS OF DIFFERENTIAL INEQUALITIES IN HILBERT SPACE

HAJIMU OGAWA[1]

Let A be an operator in a Hilbert space and let $u(t)$ be in the domain of A for each $t\in[0,\infty)$. Assuming u is strongly differentiable, Au strongly continuous and du/dt strongly piecewise continuous, all with respect to t, we define

$$Lu = \frac{du}{dt} - Au. \tag{1}$$

In the case where A is symmetric, i.e., $(Au, v) = (u, Av)$, Cohen and Lees [1] obtained lower bounds for solutions of differential inequalities of the form

$$|Lu(t)| \leqq \phi(t)\,|u(t)|. \tag{2}$$

They proved that if $\phi\in L_p(0, \infty)$ for some p with $1 \leqq p \leqq 2$, then any solution of (2) such that $u(0)\neq 0$ satisfies

$$|u(t)| \geqq Ke^{\lambda t},$$

where $K > 0$ and λ are constants depending on the solution. Assuming that A is selfadjoint, Agmon and Nirenberg [2] found a simpler

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proof of this result, as well as some extensions, by means of convexity theorems. The purpose of this paper is to present still simpler proofs, assuming only that A is symmetric, of the theorem of Cohen and Lees for $p=2$ and of the extensions of Agmon and Nirenberg.

THEOREM. *Suppose A is symmetric and let u be a solution of* (2).

(i) *If $\phi \in L_p(0, \infty)$ for some p with $2 \leqq p \leqq \infty$, then*

$$|u(t)| \geqq |u(0)| \exp[\lambda t - \mu(t+1)^{2-2/p}]. \tag{3}$$

(ii) *If $\phi(t) \leqq K(t+1)^\alpha$, $\alpha > 0$, then*

$$|u(t)| \geqq |u(0)| \exp[\lambda t - \mu(t+1)^{2\alpha+2}]. \tag{4}$$

In each case, λ is a constant depending on u, while μ is a constant depending only on ϕ.

PROOF. Assuming $|u(t)| \neq 0$ for all $t \geqq 0$, we first note from (1) and the hypothesis that A is symmetric that

$$\frac{d}{dt} \log |u|^2 = \frac{2(Au, u) + 2 \operatorname{Re}(Lu, u)}{|u|^2}. \tag{5}$$

Moreover, the strong differentiability of u and the strong continuity of Au imply

$$\frac{d}{dt}(Au, u) = 2 \operatorname{Re}\left(Au, \frac{du}{dt}\right).$$

It follows that

$$\begin{aligned} |u|^4 \frac{d}{dt} \frac{(Au, u)}{|u|^2} &= 2|u|^2 \operatorname{Re}(Au, Au + Lu) \\ &\quad - 2(Au, u) \operatorname{Re}(Au + Lu, u) \\ &= 2|Au + \tfrac{1}{2}Lu|^2 |u|^2 - \tfrac{1}{2}|Lu|^2 |u|^2 \\ &\quad - 2[\operatorname{Re}(Au + \tfrac{1}{2}Lu, u)]^2 + \tfrac{1}{2}[\operatorname{Re}(Lu, u)]^2. \end{aligned}$$

Applying Schwarz's inequality and (2) to this equation, we then find

$$\frac{d}{dt} \frac{(Au, u)}{|u|^2} \geqq -\tfrac{1}{2}\phi^2. \tag{6}$$

For the case (i), we assert that

$$\frac{(Au, u)}{|u|^2} \geqq \lambda - Mt^{1-2/p}, \tag{7}$$

where $\lambda = (Au(0), u(0))/|u(0)|^2$ and M is a constant depending only on ϕ. For $p=2$ and $p=\infty$, this is an immediate consequence of (6) For $2<p<\infty$, we make use of Hölder's inequality to obtain the estimate

$$\int_0^t \phi^2\,ds \leqq \left(\int_0^t \phi^p\,ds\right)^{2/p}\left(\int_0^t ds\right)^{1-2/p} \leqq Mt^{1-2/p},$$

so (7) follows from the integration of (6).

Applying (7) and (2) to the equation (5), we see that

$$\frac{d}{dt}\log |u(t)| \geqq \lambda - Mt^{1-2/p} - \phi(t).$$

Integrating this inequality and applying Hölder's inequality to the term in ϕ, we find that

$$\log |u(t)| \geqq \log |u(0)| + \lambda t - \frac{p}{2p-2}Mt^{2-2/p} - Nt^{1-1/p},$$

where N depends only on ϕ. Since the last two terms are bounded below by

$$-\mu(t+1)^{2-2/p}$$

for some constant μ, (3) follows.

For the case (ii), the lower bound (4) is easily verified if we integrate (6) and apply the resulting inequality, together with (2), to the equation (5).

If $|u(0)| \neq 0$, the assumption that $|u(t)| \neq 0$ for $t>0$ can easily be shown to be valid. For suppose the contrary, and let t_0 be the first point where $|u(t)|=0$. Then (3) or (4) holds for $0 \leqq t < t_0$, and by continuity the bound also holds at t_0, thus contradicting $|u(t_0)|=0$.

Bibliography

1. P. J. Cohen and M. Lees, *Asymptotic decay of solutions of differential inequalities*, Pacific J. Math. **11** (1961), 1235–1249.

2. S. Agmon and L. Nirenberg, *Properties of solutions of ordinary differential equations in Banach space*, Comm. Pure Appl. Math. **16** (1963), 121–239.

University of California, Riverside