A REMARK ON WIENER'S TAUBERIAN THEOREM

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A recent note by Levinson [1] made it seem worthwhile to point out that a weaker version of the Tauberian theorem can be proved in a few lines which is, however, strong enough to provide a proof of the prime number theorem.

Let \( K(x) \in L(-\infty, \infty) \) and assume that its Fourier transform obeys the standard condition

\[
\kappa(\xi) = \int_{-\infty}^{\infty} K(x)e^{ix\xi} \, dx
\]

\[ \neq 0 \quad \text{for all} \quad -\infty < \xi < \infty. \]

One version of Wiener's Tauberian theorem is the assertion that if \( m(y) \) is a bounded measurable function such that for almost all \( x \),

\[
\int_{-\infty}^{\infty} K(x - y)m(y) \, dy = 0
\]

then \( m(y) = 0 \) almost everywhere.

The weaker version of the Tauberian theorem is obtained by adding an extra requirement on the function \( K(x) \), namely that

\[
x^2K(x) \in L(-\infty, \infty).
\]

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To use this to prove the prime number theorem, we can follow the proof given by Levinson, since here one had

\[ K(x) = \begin{cases} 
0 & \text{for } x \leq 0 \\
R(e^s)e^{-x} & \text{for } x > 0 
\end{cases} \]

where \( R \) is a bounded function; condition (3) is thus satisfied with "plenty to spare."

To prove the weaker version, consider the class \( \Phi \) of functions \( \phi \) which have a continuous second derivative and which vanish outside a bounded interval. Let \( \phi(\xi) \in \Phi \), and set

\[ F(x) = \int_{-\infty}^{\infty} \phi(\xi)e^{ix\xi} \, d\xi. \]

Clearly, \( F(x) \in L(-\infty, \infty) \), and \( |F(x)| \cdot |K(x-y)| \cdot |m(y)| \) is integrable as a function of \((x, y)\), where \( K \) and \( m \) obey the hypotheses above. Hence, using (2) and Fubini's theorem, we have

\[ 0 = \int_{-\infty}^{\infty} F(x) \left( \int_{-\infty}^{\infty} K(x-y)m(y) \, dy \right) \, dx \\
= \int_{-\infty}^{\infty} m(y) \left( \int_{-\infty}^{\infty} K(x-y)F(x) \, dx \right) \, dy \\
\]

and clearly

\[ \int_{-\infty}^{\infty} K(x-y)F(x) \, dx = \int_{-\infty}^{\infty} \kappa(\xi)\phi(\xi)e^{ix\xi} \, d\xi. \]

Thus, for each function \( \phi \) in \( \Phi \), we will have

\[ 0 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(y)\kappa(\xi)\phi(\xi)e^{ix\xi} \, d\xi \, dy. \]

The stronger requirement (3) on \( K(x) \) implies that its transform \( \kappa(\xi) \) has a continuous second derivative; since, by assumption (1), \( \kappa(\xi) \) is never zero, we see that multiplication by \( \kappa \) carries the class \( \Phi \) into itself exactly: \( \kappa \Phi = \Phi \). We can rewrite (7) as

\[ 0 = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \phi(\xi)e^{ix\xi} \, d\xi \right) \, dy \]

for every function \( \phi \) in the class \( \Phi \). Since \( \Phi \) is closed under translation, we can replace \( \phi(\xi) \) by \( \phi(\xi-\alpha) \) and apply the usual change of variable to arrive at
(9) \[ 0 = \int_{-\infty}^{\infty} m(y) \left( \int_{-\infty}^{\infty} \phi(\xi) e^{it\xi} \, d\xi \right) e^{i\alpha y} \, dy \]

holding now for all real \( \alpha \). Using (4), this may be written as

(10) \[ 0 = \int_{-\infty}^{\infty} m(y) F(y) e^{i\alpha y} \, dy \]

for all real \( \alpha \). By the uniqueness of Fourier transforms, we may conclude that

(11) \[ m(y) F(y) = 0 \]

for almost all \( y \).

Since \( \phi \) has compact support, \( F(y) \) is an entire function, and can be chosen not to be identically zero. Since it can then have at most a denumerable number of zeros, \( m(y) = 0 \) for almost all \( y \), and the proof is complete.

It should perhaps be pointed out that the proof above uses implicitly the concept of a generalized Fourier integral (forced upon us by the fact that \( m(y) \) is merely bounded). Also, the relation \( k\Phi = \Phi \) is somewhat reminiscent of the algebraic nature of the Tauberian theorem.

Reference