ORTHOGONAL EXPANSIONS WITH POSITIVE COEFFICIENTS

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In this note we give a simple proof of a generalization of a theorem of Szegö. The theorem in question is

$$(1) \qquad (\sin\theta)^{2\lambda-1} P_n^{(\lambda)}(\cos\theta) = \sum_{k=0}^{\infty} \alpha_{k,n}^{\lambda} \sin(n+2k+1)\theta$$

where $\lambda > 0$, $\lambda \neq 1, 2, \cdots$, and

$$\alpha_{k,n}^{\lambda} = \frac{2^{2-2\lambda}(n+k)!\Gamma(n+2\lambda)\Gamma(k+1-\lambda)}{\Gamma(\lambda)\Gamma(1-\lambda)k!n!\Gamma(n+k+\lambda+1)}$$

and $P_n^{(\lambda)}(\cos\theta)$ is given by

$$(1-2r\cos\theta+r^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(\cos\theta) r^n.$$

Recall that $\sin(n+1)\theta/\sin\theta = P_n^{(1)}(\cos\theta)$ so that (1) may be written

$$(\sin\theta)^{2\lambda-1} P_n^{(\lambda)}(\cos\theta) = \sum_{k=0}^{\infty} \alpha_{k,n}^{\lambda} P_{n+2k}^{(1)}(\cos\theta)(\sin\theta)$$

or

$$(\sin\theta)^{2\lambda} P_n^{(\lambda)}(\cos\theta) = \sum_{k=0}^{\infty} \alpha_{k,n}^{\lambda} P_{n+2k}^{(1)}(\cos\theta)(\sin\theta)^2.$$

This suggests that a formula of the form

$$(2) \qquad (\sin\theta)^{2\lambda} P_n^{(\lambda)}(\cos\theta) = \sum_{k=0}^{\infty} \alpha_{k,n}^{\lambda,\mu} P_n^{(\mu)}(\cos\theta)(\sin\theta)^{2\mu}$$

is true. Since $\alpha_{k,n}^{\lambda}$ is positive for $0<\lambda<1$ we might conjecture that $\alpha_{k,n}^{\lambda,\mu}$ is positive for $\lambda<\mu$. In fact, we show that it is for $(\mu-1)/2<\lambda<\mu$. The condition $(\mu-1)/2<\lambda$ is necessary to obtain convergence of the series (2).

This result follows from an old result of Gegenbauer [1] which has almost been forgotten by the mathematical community. Since $P_n^{(\mu)}(x)$ is a polynomial we may write

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$$P_n^{(\mu)}(x) = \sum_{k=0}^{n} \beta_{k,n}^{\lambda,\mu} P_k^{(\lambda)}(x). \tag{3}$$

Gegenbauer gives the value of β as

$$\beta_{k,n}^{\lambda,\mu} = \frac{\Gamma(\lambda)(k+\lambda)\Gamma((n-k)/2+\mu-\lambda)\Gamma(\mu+(k+n)/2)}{\Gamma(\mu)[(n-k)/2]!\Gamma(\mu-\lambda)\Gamma(\lambda+1+(k+n)/2)}$$

if $n-k$ is even, $n \geqq k$, and $\beta_{k,n}^{\lambda,\mu}=0$ otherwise. Observe that $\beta \geqq 0$ if $\mu \geqq \lambda$. A simple proof of this is in [2].

We also need the following simple fact. Let $w(x)$ and $w_1(x)$ be positive functions on a set E. Let $\{p_n(x)\}$ and $\{q_n(x)\}$ be the orthonormal polynomials associated with $w(x)$ and $w_1(x)$ respectively. Then if

$$q_n(x) = \sum_{k=0}^{n} c_{k,n} p_k(x) \tag{4}$$

we have

$$w(x)p_k(x) = \sum_{n=k}^{\infty} c_{k,n} q_n(x) w_1(x). \tag{5}$$

The convergence of the series can be taken in the appropriate L^2 space if $[w(x)]^2/w_1(x)$ is integrable. (5) follows immediately from (4) since the Fourier coefficients are the same in the two expansions.

Using (3), (4) and (5) and remembering that $\{P_n^{(\lambda)}(x)\}$ are orthogonal but not orthonormal we get

$$(1-x^2)^{\lambda-1/2} P_n^{(\lambda)}(x) = \sum_{k=0}^{\infty} \alpha_{k,n}^{\lambda,\mu} P_{n+2k}^{(\mu)}(x)(1-x^2)^{\mu-1/2} \tag{6}$$

where

$$\alpha_{k,n}^{\lambda,\mu} = \frac{\Gamma(\mu)2^{2\mu-2\lambda}(n+2k+\mu)(n+2k)!\Gamma(n+2\lambda)\Gamma(n+k+\mu)\Gamma(k+\mu-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\lambda)n!k!\Gamma(n+k+\lambda+1)\Gamma(n+2k+2\mu)}.$$

Using the asymptotic formula for $P_n^{(\lambda)}(x)$ it is easy to see that the series in (6) converges for $-1<x<1$ if $\lambda>(\mu-1)/2$. Also observe that $\alpha_{k,n}^{\lambda,\mu}>0$ for $(\mu-1)/2<\lambda<\mu$.

Setting $n=0$ in (6) we get

$$\begin{aligned}(1-x^2)^{-\alpha} = {} & \frac{\Gamma(\mu)2^{2\alpha}\Gamma(2\mu-2\alpha)}{\Gamma(\alpha)\Gamma(\mu-\alpha)} \\ & \cdot \sum_{k=0}^{\infty} \frac{(2k+\mu)(2k)!\Gamma(k+\mu)\Gamma(k+\alpha)}{k!\Gamma(k+\mu-\alpha+1)\Gamma(2k+2\mu)} P_{2k}^{(\mu)}(x).\end{aligned} \tag{7}$$

For $\alpha=1/2$ and $\mu=1/2$ this is due to Bauer and $\alpha=1/2$, $\mu>0$ it is due to Gegenbauer [1].

There are two other instances of polynomial expansions of different orthogonal polynomials that involve the classical polynomials and have positive coefficients.

The better known one is

$$L_n^{(\beta)}(x) = \sum_{k=0}^{n} \frac{\Gamma(\beta-\alpha+n-k)}{\Gamma(\beta-\alpha)(n-k)!} L_k^{(\alpha)}(x), \tag{8}$$

where $L_n^{(\beta)}(x)$ are the Laguerre polynomials of order β, degree n, and $\beta>\alpha$. Notice that the coefficients are positive. See [3, p. 209] for (8).

For Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$, the following expansion holds

$$P_n^{(\beta,\gamma)}(x) = \frac{\Gamma(n+\gamma+1)}{\Gamma(\beta-\alpha)\Gamma(n+\gamma+\beta+1)} \sum_{k=0}^{n}$$

$$\cdot\frac{\Gamma(n+k+\beta+\gamma+1)\Gamma(n-k+\beta-\alpha)\Gamma(k+\alpha+\gamma+1)(2k+\alpha+\gamma+1)}{\Gamma(n+k+\alpha+\gamma+2)\Gamma(n-k+1)\Gamma(k+\gamma+1)} P_k^{(\alpha,\gamma)}(x).$$

See [4, p. 254]. Again notice that, if $\gamma>-1$ and $\beta>\alpha\geqq 0$, the coefficients are positive. The common feature of both of these results is that the polynomials are normalized correctly at the right point. In the Laguerre case $L_n^{(\alpha)}(0)>0$ and $P_n^{(\alpha,\beta)}(1)>0$ for Jacobi polynomials. An interesting conjecture can be formulated of which these are a special case. For convenience we formulate it on $(0, \infty)$.

Let $w(x)$ be a positive function on $(0, \infty)$ such that $\int_0^\infty x^n w(x)dx$ exists for each $n=0, 1, \cdots$. Let $\{p_n(x)\}$ be the orthonormal polynomials with respect to $w(x)$ normalized by $p_n(0)>0$. All of the zeros of $p_n(x)$ are in $(0, \infty)$ so this is possible. Let $\{p_n^\alpha(x)\}$ be the polynomials orthonormal with respect to $x^\alpha w(x)$ normalized in the same way. Then if $\alpha>0$ we conjecture that

$$p_n^\alpha(x) = \sum_{k=0}^{n} \alpha_k p_k(x)$$

with $\alpha_k>0$.

For α an integer this follows from known classical results. For if $\alpha=1$ then

$$c_n p_n^1(x) = \frac{p_{n+1}(0)p_n(x) - p_n(0)p_{n+1}(x)}{x}$$

by a theorem of Christoffel [4, Theorem 2.5]. By another theorem of Christoffel [4, Theorem 3.2.2],

$$\frac{p_{n+1}(0)p_n(x) - p_n(0)p_{n+1}(x)}{x} = r_n \sum_{k=0}^{n} p_k(0)p_k(x).$$

Thus

$$p_n^1(x) = r_n \sum_{k=0}^{n} p_k(0)p_k(x)$$

and $p_k(0)$, $p_n^1(0)$, and thus r_n are positive by assumption. For larger integral values of α the result follows by iteration.

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