ON THE MEAN MODULUS OF TRIGONOMETRIC POLYNOMIALS WHOSE COEFFICIENTS HAVE RANDOM SIGNS

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1. **Introduction.** Let (m_k) be an infinite sequence of distinct integers. We denote by $s_n(t, x)$ $(0<t<1,\ 0\leqq x<1)$ the nth partial sum of the trigonometric series

$$\sum_{k=1}^{\infty} \phi_k(t)c_k e(m_k x),$$

where $(\phi_k(t))$ is the system of Rademacher's functions, i.e.

$$\phi_k(t) = \operatorname{sign} \sin 2^k \pi t \qquad (k = 1, 2, \cdots),$$

and $e(x)=e^{2\pi x}$. The coefficients c_k may be complex numbers.

R. Salem and A. Zygmund [2] have investigated in detail the order of magnitude of the maximum modulus of $s_n(t, x)$,

$$\max_{0\leqq x<1} | s_n(t, x) | ,$$

for the special case of $(m_k)=(k)$, the sequence of positive integers. The purpose of this note is to present some results concerning the order of magnitude of the mean modulus

$$\int_0^1 | s_n(t, x) | \, dx,$$

where the sequence (m_k) may be arbitrary.

It is clear that for every t and every n

$$\int_0^1 | s_n(t, x) | \, dx \leqq R_n^{1/2},$$

where $R_n = \sum_{k=1}^{n} |c_k|^2$, whatever the sequence (m_k) may be. As to the lower bound for $\int_0^1 |s_n(t, x)| dx$ we shall prove the following theorems:

THEOREM 1. *Let (m_k) be an arbitrary sequence of distinct integers. Then, given any $\epsilon>0$, there exists a positive constant B_ϵ depending only on ϵ, such that except for a set of t's of measure less than ϵ we have*

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$$\int_0^1 |s_n(t,x)|\,dx \geqq B_t R_n^{1/2} \tag{1}$$

for all $n \geqq 1$.

We write for the sake of brevity

$$S_n = \sum |c_i|^2 |c_j|^2 |c_k|^2 |c_l|^2,$$

the summation being extended over all indices i, j, k, l with $1 \leqq i, j, k, l \leqq n$ such that $m_i + m_j = m_k + m_l$ (order is relevant).

THEOREM 2. *Let* (m_k) *be an arbitrary sequence of distinct integers. If* $S_n/R_n^4 = O(n^{-\alpha})$ *for some* $\alpha > 1$, *then we have*

$$\liminf_{n\to\infty} R_n^{-1/2} \int_0^1 |s_n(t,x)|\,dx \geqq 2^{-1/2} \tag{2}$$

almost everywhere in t.

The condition imposed on S_n/R_n^4 in Theorem 2 obviously implies that R_n tends to infinity with n. Note that for any sequence (m_k) of distinct integers we have always $1 \leqq \alpha \leqq 2$, if $S_n/R_n^4 = O(n^{-\alpha})$; in the case of $\alpha = 1$, which is excluded from Theorem 2, one may also prove the validity of (2), assuming that the growth of R_n as $n \to \infty$ is sufficiently regular (see §3 below).

2. **Proof of the theorems.** We have

$$\int_0^1 |s_n(t,x)|^2 dx = \sum_{k=1}^n \phi_k^2(t) |c_k|^2 = R_n$$

almost everywhere in t. Hence we obtain by Hölder's inequality

$$\begin{aligned} R_n &= \int_0^1 |s_n(t,x)|^2\,dx \\ &\leqq \left(\int_0^1 |s_n(t,x)|\,dx\right)^{2/3} \left(\int_0^1 |s_n(t,x)|^4\,dx\right)^{1/3} \end{aligned} \tag{3}$$

for almost all t, where, as is readily seen,

$$\int_0^1 dt \int_0^1 |s_n(t,x)|^4 dx = 2R_n^2 - T_n, \qquad T_n = \sum_{k=1}^n |c_k|^4. \tag{4}$$

Let E denote the set of t, $0 < t < 1$, for which the integral

$$\int_0^1 |s_n(t, x)|^4 dx \geqq 2R_n^2 - T_n + A.$$

Then it follows from (4) that the measure $m(E)$ of E satisfies

$$m(E) \leqq \frac{2R_n^2 - T_n}{2R_n^2 - T_n + A} \leqq \frac{2}{2 + a} < \epsilon,$$

if we put $A = aR_n^2$ and take $a > 0$ sufficiently large. Thus, for any $t \notin E$ we have

$$\int_0^1 |s_n(t, x)|^4 dx < (2 + a)R_n^2,$$

and we obtain, via (3),

$$\int_0^1 |s_n(t, x)| \, dx \geqq \frac{R_n^{3/2}}{(2 + a)^{1/2} R_n} = B_\epsilon R_n^{1/2}$$

with $B_\epsilon = (2+a)^{-1/2}$. This is (1).

Now, let us consider the integral

$$I_n = \int_0^1 R_n^{-4} \left(\int_0^1 |s_n(t, x)|^4 dx - 2R_n^2 + T_n \right)^2 dt.$$

It is not difficult to verify that

$$\int_0^1 \left(\int_0^1 |s_n(t, x)|^4 dx \right)^2 dt = (2R_n^2 - T_n)^2 + O(S_n).$$

By the assumption $S_n/R_n^4 = O(n^{-\alpha})$ $(\alpha > 1)$ and the relation (4) we thus have $I_n = O(n^{-\alpha})$ and therefore $\sum_1^\infty I_n < \infty$. Hence

$$\sum_{n=1}^{\infty} R_n^{-4} \left(\int_0^1 |s_n(t, x)|^4 dx - 2R_n^2 + T_n \right)^2 < \infty$$

for almost all t. It follows in particular that for almost all t and any $\epsilon > 0$ we have for all $n \geqq n_0(t, \epsilon)$

$$\int_0^1 |s_n(t, x)|^4 dx < (2 + \epsilon)R_n^2$$

so that

$$R_n^{-1/2}\int_0^1 |s_n(t, x)|\,dx > (2+\epsilon)^{-1/2}$$

by (3) again. Since $\epsilon > 0$ is arbitrary, this proves (2).

3. **Remarks.** (1) Let us consider the case of $S_n/R_n^4 = O(1/n)$. It will be immediately clear from the argument of §2 that we have

$$\sum_{m=1}^{\infty} I_{m^2} < \infty,$$

so that

$$\liminf_{m\to\infty} R_{m^2}^{-1/2}\int_0^1 |s_{m^2}(t, x)|\,dx \geqq 2^{-1/2}$$

for almost all t.

Suppose now that

$$R_{(m+1)^2}/R_{m^2} \to 1 \quad \text{as} \quad m \to \infty.$$

Let n be any integer between m^2 and $(m+1)^2$. Then from the inequality

$$\left|\int_0^1 |s_n(t, x)|\,dx - \int_0^1 |s_{m^2}(t, x)|\,dx\right| \leqq \int_0^1 |s_n(t, x) - s_{m^2}(t, x)|\,dx$$
$$\leqq (R_n - R_{m^2})^{1/2}$$

it follows that

$$R_n^{-1/2}\int_0^1 |s_n(t, x)|\,dx \geqq \left(\frac{R_{(m+1)^2}}{R_{m^2}}\right)^{-1/2} R_{m^2}^{-1/2}\int_0^1 |s_{m^2}(t, x)|\,dx$$
$$- \left(\frac{R_{(m+1)^2}}{R_{m^2}} - 1\right)^{1/2}.$$

Hence

$$\liminf_{n\to\infty} R_n^{-1/2}\int_0^1 |s_n(t, x)|\,dx \geqq 2^{-1/2}$$

for almost all t.

As an example we take $c_k = 1$ $(k = 1, 2, \cdots)$. We have then $R_n = T_n = n$, and $S_n = O(n^3)$ for any sequence (m_k) of distinct integers. Therefore

$$\liminf_{n\to\infty} n^{-1/2} \int_0^1 \left| \sum_{k=1}^{n} \phi_k(t) e(m_k x) \right| dx \geqq 2^{-1/2}$$

almost everywhere in t.

(2) It is obvious that our Theorems 1 and 2 have analogues for partial sums $t_n(t, x)$ of real trigonometric series

$$\sum_{k=1}^{\infty} \phi_k(t) a_k \cos 2\pi(m_k x - \alpha_k)$$

with real coefficients a_k and phases α_k. Put

$$P_n = \frac{1}{2} \sum_{k=1}^{n} a_k^2$$

and

$$Q_n = \sum a_i^2 a_j^2 a_k^2 a_l^2,$$

where the summation is taken over $1 \leqq i, j, k, l \leqq n$ such that

$$\pm m_i \pm m_j \pm m_k \pm m_l = 0.$$

Then, given any $\epsilon > 0$, we have for all t but a set of measure less than ϵ and all $n \geqq 1$

$$\int_0^1 | t_n(t, x) | \, dx \geqq C_\epsilon P_n^{1/2}$$

with some constant $C_\epsilon > 0$, and, if $\beta > 1$ or $\beta = 1$ and $P_{(m+1)^2}/P_{m^2} \to 1$ as $m \to \infty$, where $Q_n/P_n^4 = O(n^{-\beta})$, we have for almost all t

$$\liminf_{n\to\infty} P_n^{-1/2} \int_0^1 | t_n(t, x) | \, dx \geqq 3^{-1/2}.$$

The proof of these results is quite similar to that of the results for complex polynomials $s_n(t, x)$.

(3) Let $m_1, \cdots, m_n$ be any set of n distinct integers. S. Chowla has conjectured that

$$\min_{0 \leqq x < 1} \sum_{k=1}^{n} \cos 2\pi m_k x < -Cn^{1/2}$$

for some absolute constant $C > 0$ (cf. [1]). If this conjecture is true, it is essentially the best possible.

We can show that, given n distinct integers $m_1, \cdots, m_n$, there exists always a subset $m_{i_1}, \cdots, m_{i_r}$ of $m_1, \cdots, m_n$ for which

$$\min_{0 \leq x < 1} \sum_{j=1}^{r} \cos 2\pi m_{i_j} x < -\frac{1}{4}\left(\frac{n}{6}\right)^{1/2}.$$

This is an easy consequence of the fact that

$$\int_0^1 \left| \sum_{k=1}^{n} \epsilon_k \cos 2\pi m_k x \right| dx > \left(\frac{n}{6}\right)^{1/2}$$

for some sequence (ϵ_k) of ± 1, which is a particular case of the results mentioned in (2).

References

1. P. Erdös, *Some unsolved problems*, Publ. Math. Inst. Hungar. Acad. Sci. 6 Ser. A (1961), 221–254 (especially, pp. 247–248).

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