LOCAL CONVEXITY AND L_n SETS[1]

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We will prove that a closed connected set S in E_r whose points of local nonconvexity can be decomposed into a finite number of convex parts also possesses a simple type of polygonal connectedness. Furthermore, if S has a finite number of points of local nonconvexity, we obtain another result. In order to describe this situation easily we use the notation of L_n set used by Horn and Valentine [1] in 1949. For other results related to L_n sets, see A. Bruckner and J. Bruckner [1] and Valentine [1]. The following notations and definitions will enable us to express the results more readily.

Notations. The interior, boundary and closure of a set S in Euclidean r-space E_r are denoted by int S, bd S and cl S respectively. If $x \in S$, $y \in S$, then xy denotes the closed line segment joining x and y. The symbols $\cup$, $\cap$ and $\sim$ denote set union, set intersection and set difference respectively. The symbol 0 is used to denote the empty set. The convex hull of a set $S \subset E_r$ is indicated by conv S.

DEFINITION 1. A set S is called an L_n set if each pair of points in S can be joined by a polygonal arc of S containing at most n segments.

DEFINITION 2. A point $x \in S$ is a point of local convexity of S if there exists a neighborhood N of x such that $N \cap S$ is convex; otherwise, x is called a point of local nonconvexity of S.

DEFINITION 3. A set S is starshaped relative to a point p if for every $x \in S$ it is true that $px \subset S$.

The following two theorems together with Theorem 3 stated later contain the main results of this paper. The symbols n and r always denote non-negative integers.

THEOREM 1. *Suppose S is a closed connected set in E_r which has at most n points of local nonconvexity.*

Then S is an L_{n+1} set.

THEOREM 2. *Suppose S is a closed connected set in E_r. Furthermore, suppose that the set Q of all points of local nonconvexity of S can be decomposed into n convex subsets.*

Then S is an L_{2n+1} set.

It should be observed that if $n=0$ in Theorems 1 and 2, then each

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reduces to the important theorem of Tietze [1] which we state as follows, since it will be used.

THEOREM (TIETZE [1]). *If S is a closed connected set in E_r all of whose points are points of local convexity, then S is convex.*

COROLLARY 1 (TO TIETZE'S THEOREM). *Let S be a closed set in E_r. Suppose $x\in S$, $y\in S$, $z\in S$ with $xy\subset S$, $yz\subset S$. If the triangular set $\Delta\equiv \mathrm{conv}(x\cup y\cup z)$ determined by x, y, z contains no points of local nonconvexity of S, then $\Delta\subset S$.*

PROOF. Corollary 1 follows from Tietze's Theorem since that component of $S\cap\Delta$ which contains xy and yz is a closed connected set having no points of local nonconvexity.

COROLLARY 2 (TO COROLLARY 1). *Let S be a closed set in E_r. Suppose $x\in S$, $y\in S$ with $xy\subset S$, $yz\subset S$. If the only possible points of local nonconvexity of S in $\Delta\equiv \mathrm{conv}(x\cup y\cup z)$ are x and z, then $\Delta\subset S$.*

PROOF. Choose points $x_i\in xy$, $z_i\in yz$ so that $\mathrm{conv}(x_i\cup y\cup z_i)$ contains no points of local nonconvexity of S. Corollary 1 implies $\mathrm{conv}(x_i\cup y\cup z_i)\subset S$. Since S is closed, an obvious limit procedure implies $\Delta\subset S$.

The following lemma will shorten the proofs of Theorems 1 and 2.

LEMMA 1. *Suppose S is a closed connected set which has at least one point of local nonconvexity. If x is a point of S, then there exists at least one point p of local nonconvexity of S such that $xp\subset S$.*

PROOF. Let Q denote the collection of all points of local nonconvexity of S. Since S is closed, Definition 2 implies that Q is closed. Firstly, if $x\in Q$, Lemma 1 follows by choosing $p=x$, since $xp\subset S$. Secondly, if $x\notin Q$, let $S(x)$ denote the set of all points of S which can see x via S, so that

$$S(x) = \{y: xy\subset S\}. \tag{1}$$

We wish to prove that $Q\cap S(x)\neq 0$. To do this, suppose $Q\cap S(x)=0$. We will prove that this implies $S(x)$ is both open and closed in S, so that $S(x)=S$, S being connected. To prove $S(x)$ is open relative to S, choose $y\in S(x)$. Since Q is closed, condition (1) and $Q\cap S(x)=0$ imply there exists a neighborhood $N(y)$ of y (with $S\cap N(y)$ convex) such that if $z\in S\cap N(y)$, then $Q\cap\mathrm{conv}(x\cup y\cup z)=0$. Corollary 1 then implies that $xz\subset S$. Hence, $S\cap N(y)\subset S(x)$ so that $S(x)$ is *open* relative to S. To prove that S is also *closed* relative to S let $y_j\rightarrow y$ ($j=1, 2, \cdots$) where $y_j\in S(x)$. Then, since $xy_j\subset S(x)$, and since S is closed, we have $xy\subset S$ so that $y\in S(x)$. Hence $S(x)=S$, since S is connected. This and the assumption $Q\cap S(x)=0$ imply $S\cap Q=0$. Since $S\neq 0$, we have $Q=0$, a contradiction, because the cardinality of Q is greater than zero, by assumption. Thus, $Q\cap S(x)\neq 0$, and Lemma 1 has been proved.

Finally, before proving Theorems 1 and 2, we state the following useful concept.

DEFINITION 4. Let Q be a subset of a set S in E_r. An s-fold connected convex covering $\mathcal{P}$ for Q is a collection of s or fewer closed convex sets whose union is a connected subset of S which contains Q.

PROOF OF THEOREM 2. First, observe that if $n=0$, then Tietze's Theorem [1] implies Theorem 2. Hence, suppose $n\geqq 1$. Our hypotheses imply that the collection Q of all points of local nonconvexity can be decomposed into n *closed convex* subsets which we designate by $Q_1, \cdots, Q_n$, so that

$$Q = \bigcup_{i=1}^{n} Q_i.$$

The sets Q_i $(i=1, \cdots, n)$ may be taken to be closed and convex since S is closed. They may or may not be disjoint. We will prove that there exists a $(2n-1)$-fold connected convex covering $\mathcal{P}$ for Q as described in Definition 4. If $n=1$, let $\mathcal{P}=\{Q\}$, since Q is a 1-fold connected convex covering of itself. Hence, suppose $n>1$. Since each set Q_i is its own 1-fold connected convex covering, the collection $\{Q_1, Q_2, \cdots, Q_n\}$ has a maximal subcollection of s members, which, relative to S, has a $(2s-1)$-fold connected convex covering. Without loss of generality we will rearrange subscripts so that this maximal subcollection is

$$\mathcal{P}_1 = \{Q_1, Q_2, \cdots, Q_s\}, \qquad s \leqq n.$$

Remember that $\bigcup_{i=1}^{s} Q_i$ has a $(2s-1)$-fold connected convex covering in S. We will prove that $s=n$. To do this suppose $s<n$, and let

$$\mathcal{P}_2 \equiv \{Q_{s+1}, Q_{s+2}, \cdots, Q_n\}.$$

Also let

$$Q^1 \equiv \bigcup_{i=1}^{s} Q_i, \qquad Q^2 \equiv \bigcup_{i=s+1}^{n} Q_i. \tag{2}$$

The maximality of $\mathcal{P}_1$ as described above implies $Q^1\cap Q^2=0$. Lemma 1 implies that for each point $x\in S$ there exists either a point $y_1\in Q^1$ or a point $y_2\in Q^2$ such that at least one of the two segments xy_1, xy_2 is in S. Since S is a closed connected set, the preceding sentence implies the existence of at least one point $y \in S$ and a pair of points $y_1 \in Q^1$, $y_2 \in Q^2$ such that y_i ($i=1, 2$) is a *nearest* point of Q^i which satisfies the condition

$$(3) \qquad yy_i \subset S \qquad (i = 1, 2).$$

Hence, $|yy_i|$ = the distance between y and $S(y) \cap Q^i$. Let Q_j and Q_k denote elements of $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively such that

$$(4) \qquad y_1 \in Q_j, \qquad y_2 \in Q_k.$$

First, if $y \in Q^1$ then $Q^1 \cup Q_k$ would have a $[2(s+1)-1]$-fold connected convex covering consisting of the $(2s-1)$-fold connected convex covering for Q^1 together with y_1y_2 and Q_k. Hence $\mathcal{P}_1$ would not be maximal, so that $y \notin Q^1$. In exactly the same way we have $y \notin Q^2$. Hence, $y \notin Q$. Now, since y_i is a nearest point of Q^i to y which satisfies (3), the maximality of $\mathcal{P}_1$ implies

$$(5) \qquad Q \cap yy_i = y_i \qquad (i = 1, 2).$$

Without loss of generality we may assume that

$$(6) \qquad |yy_1| \geqq |yy_2|.$$

Since $Q^2 \cap yy_1 = 0$, let $z \in yy_2$ be that point such that $y_1z \cap Q^2 \neq 0$, and such that $\text{conv}(y \cup y_1 \cup z)$ contains no points of Q^2 in its interior. Since $y \notin Q$, no points of $Q^1 \sim y_1$ are in the $\text{conv}(y \cup y_1 \cup z)$, otherwise there will exist a point $y^* \in S(y) \cap Q^1$ with $|yy^*| < |yy_1|$, thus violating (3). Let y_3 be the point of $Q^2 \cap y_1z$ nearest to y_1. Corollary 2 applied to y_1, y_3 and y implies $y_1y_3 \subset S$. Let Q_m be a member of $\mathcal{P}_2$ containing y_3. Since $y_1y_3 \subset S$, the collection Q^1 together with Q_m would have a $(2s+1)$-fold connected convex covering which violates the maximality of $\mathcal{P}_1$. Hence, we have a contradiction, and hence Q has a $(2n-1)$-fold connected convex covering.

To complete the proof, choose any pair of points x and y in S. By Lemma 1 there exists two points x_1 and x_2 of Q such that $xx_1 \subset S$, $yx_2 \subset S$. Since Q has a $(2n-1)$-fold connected convex covering x_1 and x_2 can be joined by a polygonal path in S containing at most $2n-1$ segments. Therefore, x and y can be joined by a polygonal path in S containing at most $2n+1$ segments, so that S is an L_{2n+1} set.

Proof of Theorem 1. The proof of Theorem 1 has the same form as that of Theorem 2 with the following difference. The set Q of points of local nonconvexity of S has an $(n-1)$-fold connected convex covering, and the proof of this fact can be constructed as in the proof of Theorem 2. It should be observed that in this construction the convex covering is the union of line segments with endpoints in Q if

$n>1$ and a single point if $n=1$. The fact that S is an L_{n+1} set then follows immediately from Lemma 1.

Theorem 2 can be generalized as follows. We state this result last because of its more complicated structure.

THEOREM 3. *Let S be a closed connected set in E_r. Suppose the set Q of all points of local nonconvexity of S can be decomposed into n connected closed subsets $Q_1, Q_2, \cdots, Q_n$ such that if $x\in Q_i$, $y\in Q_i$, $Q_i \in \{Q_1, Q_2, \cdots, Q_n\}$, then $xy\subset S$.*

Then S is an L_{2n+1} set.

PROOF. Just as in the proof of Theorem 2 it can be proved that the set Q can be covered by a connected set which is the union of $2n-1$ subsets $P_1, P_2, \cdots, P_{2n-1}$ such that if $x\in P_i$, $y\in P_i$, $P_i \in \{P_1, P_2, \cdots, P_{2n-1}\}$, then $xy\subset S$. Lemma 1 again implies that S is an L_{2n+1} set.

REMARKS. It is a relatively simple matter to construct examples in E_3 which satisfy the hypotheses of Theorem 2 and which are not L_{2n} sets, so that the theorem for $r>2$ in one respect is a best theorem. One such example consists of the appropriate union of polyhedra in E_3 each of whose points of local nonconvexity consists of a single segment, together with a suitable additional number of line segments intersecting each of these polyhedra in a unique point, the latter being a point of local nonconvexity. Since the converse of Theorem 2 is false, the theorem does not characterize L_{2n+1} sets. Corresponding remarks also apply to Theorem 1. Furthermore, it should be observed that Theorem 2 is of chief interest in E_r with $r\geqq 3$, because for sets S in the plane E_2 the hypotheses of Theorem 2 imply the hypotheses of Theorem 1, so that S is an L_{n+1} set, a stronger result. However, for $r>2$ this no longer holds, and the conclusion of Theorem 2 cannot be improved.

Questions. It appears that Theorem 2 can be improved if S is a polyhedron or if S is the closure of an open connected set. In fact the classification of points of local nonconvexity for polyhedra is itself of independent interest. In E_2, the set Q for a simple closed polygonal region are the points at which the polygonal boundary has re-entrant angles, and there are only a finite number of these so that Theorem 1 applies. Also it is a simple matter to construct polygonal regions in E_2 having exactly n points of local nonconvexity which are not L_n sets, so that Theorem 1 is a best theorem in one sense. The corresponding result for polyhedra in E_3 is still unsettled, since the points of local nonconvexity for a polyhedron may consist of a complicated combination of *line segments*. Both Theorems 2 and 3 do shed a good deal of light on this more complicated situation. However, there is still more work to be done for polyhedra in E_r, $r \geqq 3$. As a final observation note that if there exists a point $p \in S$ such that for each $y \in Q$ we have $py \subset S$, then Lemma 1 implies that S is an L_4 set. It appears that this result might have further generalizations.

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