ON THE DE LA VALLÉE POUSSIN DERIVATIVE[1]

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Let f be a real valued function over D, a subset of the reals. For any integer $n>0$ and any set $P\subset D$ consisting of $n+1$ distinct points $p_1, \cdots, p_{n+1}$, the nth divided difference of f corresponding to P, $V_n[f\colon P]$, is given by

$$V_n[f\colon P] = V_n[f\colon p_1, \cdots, p_{n+1}]$$
$$= \sum_{j=1}^{n+1} f(p_j)/[(u-p_1)\cdots(u-p_{n+1})]'_{u=p_j},$$

the "prime" denoting ordinary differentiation.

Now let D be the interval (a, b) and f continuous over D. Denote by E the set consisting of the $n+1$ points $x_1, \cdots, x_{n+1}$. If

$$\lim_{h\to 0} n!V_n[f\colon x+x_1h, \cdots, x+x_{n+1}h]$$

exists and is finite, it is called, following Denjoy [1, p. 305], the *nth E-generalized derivative* of f at x, $f_{n,E}(x)$. If $f_{n,E}(x)$ is independent of the choice of $E\subset S$ for a subset S of the reals, we will call it the *nth S-generalized derivative* $f_{n,S}(x)$.

Let f and D be as above. If for $x\in(a, b)$ we have

$$f(x+h) = a_0 + a_1h + \cdots + a_nh^n/n! + o(h^n), \tag{1}$$

where the numbers $a_i=a_i(x)$, $i=0, \cdots, n$, are independent of h, then a_n will be called, following Marcinkiewicz and Zygmund [2, p. 1], the *nth de la Vallée Poussin derivative* of f at x, $f_{(n)}(x)$.

If $f_{(n)}(x)$ exists so does $f_{(i)}(x)$ for $i=1, \cdots, n-1$. If f is differentiable n times at x, $f_{(i)}(x)$ exists for $i=1, \cdots, n$ and is equal to the ordinary derivative of the corresponding order. The converse is not true for $n\geqq 2$. Let $f(x)=e^{-x^{-2}}\sin e^{x^{-2}}$ for $x\neq 0$, and $f(0)=0$. Then, $f_{(n)}(0)=0$ for $n=1, 2, 3, \cdots$. However, the ordinary derivative of f of order at least 2 does not exist at $x=0$.

Denjoy shows [1, p. 289] that if $f_{(n)}(x)$ exists then $f_{n,S}(x)$ exists for S arbitrary and $f_{n,S}(x)=f_{(n)}(x)$. Since the existence of $f_{(n)}(x)$ implies that of $f_{(i)}(x)$, $i=1, \cdots, n-1$, it follows that $f_{i,S_i}(x)$ exists and $f_{i,S_i}(x)=f_{(i)}(x)$ for S_i arbitrary. Further, we can deduce easily from relation (1) that

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$$\lim_{h\to 0}\left\{(y-x-x_ih)V_{n+1}[f: x+x_1h,\cdots,x+x_{n+1}h, y]\right\} = \theta(x, y-x),$$
$$i = 1, \cdots, n+1,$$

where $\theta(x, y-x)\to 0$ as $y\to x$, and $x_1, \cdots, x_{n+1}$ is any $(n+1)$-tuple of points of S.

In this paper we obtain a converse of the above Denjoy theorem.

Let f and E be as above. Let $y_i = x+x_ih$ for $x\in(a, b)$. Set $E_i = \{x_1, \cdots, x_i\}$ and $Y_i = \{y_1, \cdots, y_i\}$ for $i=1, \cdots, n+1$.

THEOREM. *If*

(i) $f_{j,E_{j+1}}(x)$ *exists for all* $j=1, \cdots, n$,

(ii) $\lim_{h\to 0}\{(y-y_{n+1})V_{n+1}[f: y_1, \cdots, y_{n+1}, y]\} = \theta(x, y-x)$ *where* $\theta(x, y-x)\to 0$ *as* $y\to x$, *then* $f_{(n)}(x)$ *exists and* $f_{(n)}(x) = f_{n,E}(x)$.

PROOF. By Newton's interpolation formula we have

$$\begin{aligned} f(y) = f(y_1) &+ (y-y_1)V_1[f: Y_2] + \cdots \\ &+ (y-y_1)\cdots(y-y_n)V_n[f: Y_{n+1}] \\ &+ (y-y_1)\cdots(y-y_n)\{(y-y_{n+1})V_{n+1}[f: y_1, \cdots, y_{n+1}, y]\}. \end{aligned}$$

For $h\to 0$, we obtain

$$f(y) = f(x) + \cdots + (y-x)^n f_{n,E}(x)/n! + (y-x)^n\theta(x, y-x).$$

Taking into account relation (1), the above result implies that $f_{(n)}(x)$ exists and $f_{(n)}(x) = f_{n,E}(x)$.

REMARK. Under the assumption that $f_{n,E}(x)$ exists, the 2nd condition of our theorem is equivalent to

$$\lim_{h\to 0} V_n[f: y_1, \cdots, y_n, y] = (f_{n,E}(x)/n!) + \theta(x, y-x).$$

This follows directly from the relation

$$\begin{aligned} V_n[f: y_1, \cdots, y_n, y] &- (y-y_{n+1})V_{n+1}[f: y_1, \cdots, y_{n+1}, y] \\ &= V_n[f: y_1, \cdots, y_{n+1}]. \end{aligned}$$

REFERENCES

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