ON THE CHARACTERISTIC ROOTS OF MATRICES[1]

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Let $A=(a_{ij})$ be an $n\times n$ complex matrix. For $i=1, 2, \cdots, n$ and $p>0$ define $R(i, p)=(\sum_{t=1;t\neq i}^{n}|a_{it}|^{p})^{1/p}$. It is well known [2] that the characteristic roots of A lie in the union of the disks

$$(1)\qquad |a_{ii}-z| \leqq R(i, 1) \qquad (i=1, 2, \cdots, n).$$

A. Brauer [3] improved this result by replacing these disks with the $n(n-1)/2$ ovals of Cassini

$$(2)\qquad |a_{ii}-z||a_{jj}-z| \leqq R(i, 1)R(j, 1) \quad (i, j=1, 2, \cdots, n; i\neq j).$$

A. Ostrowski [4] extended (1) as follows:

THEOREM A. *Let $k_1, k_2, \cdots, k_n$ be positive numbers such that*

$$(3)\qquad \sum_{i=1}^{n}(k_i+1)^{-1}\leqq 1.$$

Let p and q be chosen so that

$$(4)\qquad 1/p+1/q=1 \quad \text{and} \quad p, q>1.$$

Then the characteristic roots of A lie in the union of the disks

$$(5)\qquad |a_{ii}-z| \leqq k_i^{1/q}R(i, p) \qquad (i=1, 2, \cdots, n).$$

In this paper we improve Theorem A in the same way that Brauer improved (1). It is also shown that our result generalizes Brauer's theorem (2).

THEOREM. *Let $A=(a_{ij})$ be an $n\times n$ complex matrix. Let $k_1, k_2, \cdots, k_n$, p and q be positive numbers satisfying* (3) *and* (4). *Then the characteristic roots of A lie in the union of the $n(n-1)/2$ ovals of Cassini*

$$(6)\qquad |a_{ii}-z||a_{jj}-z| \leqq (k_ik_j)^{1/q}R(i, p)R(j, p) \qquad (i, j=1, 2, \cdots, n; i\neq j).$$

We first prove the following lemma:

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LEMMA. *Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ $(n \geqq 2)$ be non-negative numbers such that*

$$\sum_{i=1}^{n} b_i \leqq \sum_{i=1}^{n} a_i = 1. \tag{7}$$

Then there exist two integers u and v such that

$$a_u a_v(1 - b_u - b_v) \geqq b_u b_v(1 - a_u - a_v).$$

PROOF. By (7) we may pick u so that $b_u \leqq a_u$. If the lemma is false, we may assume that

$$a_u a_j(1 - b_u - b_j) < b_u b_j(1 - a_u - a_j) \qquad (j = 1, 2, \cdots, n; j \neq u).$$

Hence

$$a_j(1 - b_u) < b_j(1 - a_u) \qquad (j = 1, 2, \cdots, n; j \neq u).$$

By (7) we have

$$a_j \sum_{t \neq u} b_t < b_j \sum_{t \neq u} a_t \qquad (j = 1, 2, \cdots, n; j \neq u).$$

Adding these inequalities, we obtain the contradiction

$$\sum_{j \neq u} a_j \sum_{t \neq u} b_t < \sum_{j \neq u} b_j \sum_{t \neq u} a_t.$$

PROOF OF THE THEOREM. Let θ be a characteristic root of A, with a corresponding characteristic vector $(x_1, x_2, \cdots, x_n)$ normalized so that

$$\sum_{i=1}^{n} | x_i |^2 = 1. \tag{8}$$

From the equations

$$\sum_{j=1}^{n} (a_{ij} - \theta\delta_{ij})x_j = 0 \qquad (i = 1, 2, \cdots, n)$$

(where δ is the Kronecker symbol), it follows that

$$| a_{ii} - \theta | | x_i | \leqq \sum_{t=1; t \neq i}^{n} | a_{it} | | x_t | \qquad (i = 1, 2, \cdots, n).$$

Hence

$$
\begin{aligned}
&|a_{ii}-\theta||a_{jj}-\theta||x_i||x_j| \leqq \sum_{t=1;\,t\neq i}^{n} |a_{it}||x_t| \sum_{s=1;\,s\neq j}^{n} |a_{js}||x_s| \\
&\qquad\qquad\qquad\qquad\qquad\qquad (i, j = 1, 2, \cdots, n;\ i \neq j).
\end{aligned} \tag{9}
$$

Applying Hölder's inequality [1, p. 19] and equation (8), we obtain

$$
\begin{aligned}
&|a_{ii}-\theta|^q|a_{jj}-\theta|^q|x_i|^q|x_j|^q \\
&\qquad \leqq R(i, p)^q R(j, p)^q (1 - |x_i|^q)(1 - |x_j|^q) \\
&\qquad\qquad\qquad\qquad\qquad (i, j = 1, 2, \cdots, n;\ i \neq j).
\end{aligned} \tag{10}
$$

Now we may assume that for each i, $|x_i|^q \neq 1$, since otherwise $\theta = a_{ii}$ for some i and then clearly θ lies inside the ovals (6). Letting $|x_i|^q = a_i$ and $(k_i+1)^{-1} = b_i$ for $i = 1, 2, \cdots, n$, we see by the lemma that there exist integers u and v for which

$$
\begin{aligned}
&|x_u|^q|x_v|^q(1 - (k_u+1)^{-1} - (k_v+1)^{-1}) \\
&\qquad \geqq (k_u+1)^{-1}(k_v+1)^{-1}(1 - |x_u|^q - |x_v|^q).
\end{aligned}
$$

It follows that

$$
|x_u|^q|x_v|^q k_u k_v \geqq (1 - |x_u|^q)(1 - |x_v|^q). \tag{11}
$$

Since the right side of (11) is positive, $|x_u|^q|x_v|^q > 0$. Thus (10) and (11) imply

$$
|a_{uu}-\theta||a_{vv}-\theta| \leqq (k_u k_v)^{1/q} R(u, p) R(v, p),
$$

and the theorem is proved.

Consideration of the case $\theta = 0$ in the theorem leads to the following corollary:

COROLLARY 1. *The $n \times n$ matrix A is nonsingular provided that positive numbers $k_1, k_2, \cdots, k_n$, p and q satisfying* (3) *and* (4) *can be found such that*

$$
|a_{ii}||a_{jj}| > (k_i k_j)^{1/q} R(i, p) R(j, p)
$$

for each pair $i, j = 1, 2, \cdots, n$; $i \neq j$.

Consideration of the limiting case where p approaches 1 and q becomes infinite yields Brauer's theorem (2). If we reverse this procedure and let p become infinite while q approaches 1, we obtain the following corollary:

COROLLARY 2. *Let $A = (a_{ij})$ and $k_1, k_2, \cdots, k_n$ be as above. Let*

$m_i = \operatorname{Max}_{t\neq i} |a_{it}|$, $i=1, 2, \cdots, n$. *Then the characteristic roots of A lie in the union of the ovals*

$$|a_{ii} - z|\,|a_{jj} - z| \leqq k_i k_j m_i m_j \qquad (i, j = 1, 2, \cdots, n;\ i \neq j).$$

Proof. Since $R(i, p) \leqq (m_i^p(n-1))^{1/p} = m_i(n-1)^{1/p}$, the result is immediate.

From Corollary 2 we obtain the following nonsingularity condition:

Corollary 3. *The $n\times n$ matrix $A=(a_{ij})$ is nonsingular provided that positive numbers $k_1, k_2, \cdots, k_n$ satisfying* (3) *can be found such that*

$$|a_{ii}|\,|a_{jj}| > k_i k_j m_i m_j$$

for each pair $i, j = 1, 2, \cdots, n$; $i\neq j$.

References

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