ON BERGMAN'S KERNEL FUNCTION FOR SOME UNIFORMLY ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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1. We present here a generalization of the theory of Bergman's kernel function for uniformly elliptic partial differential equations of the divergence type

$$\mathfrak{M}u \equiv \partial/\partial x_k\,(a_{ik}\partial u/\partial x_i) = 0.$$

It is known that for regular open sets Ω in $\boldsymbol{R}^n$ the expression

$$M_\Omega(u) = \int_\Omega a_{ik}\,\partial u/\partial x_i\,\partial u/\partial x_k\,dX$$

is a natural norm on the space of regular solutions of $\mathfrak{M}u=0$ vanishing at a point $x_0\in\Omega$. It is proved that for E compact in Ω, $x\in E$

$$|\,u(x)\,|^2 \leqq K(E)M_\Omega(u).$$

The existence of Bergman's kernel $K(x, y)$ and the convergence of its expansion in terms of a complete orthonormal set of functions follows at once. We prove the boundedness of $K(x, y)$ on compact subsets of Ω. A sharp value for $K(E)$ is found to be $\sup_E K(x, x)$.

2. We consider partial differential equations of the type

$$\mathfrak{M}u \equiv \partial/\partial x_k\,(a_{ik}\partial u/\partial x_i) = 0, \qquad i, k = 1, \cdots, n,$$

where the coefficients $a_{ik}\in C^{(1,\lambda)}$ in a regular region $\Omega\subset\boldsymbol{R}^n$. Moreover, $\mathfrak{M}$ satisfies a uniform ellipticity condition

$$\lambda^{-1}\sum_{i=1}^n \xi_i^2 \leqq a_{ik}\xi_i\xi_k \leqq \lambda\sum_{i=1}^n \xi_i^2.$$

We recall the definition of regularity: let Ω be a subregion of a region $V\subset\boldsymbol{R}^n$. Let B be the open unit ball centered at the origin and let P be the hyperplane $x_n=0$. Ω shall be called a *regular subregion* [1] if:

(I) Bd Ω is compact in V,

(II) every $x\in$ Bd Ω has a neighborhood $N(x)$ and a diffeomorphism h: $N(x)\to B$ such that $h(N(x)\cap\text{Bd }\Omega)=B\cap P$ and $h(N(x)\cap\Omega)$ is one

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of the two half balls of $B-P$,

(III) $\bar{\Omega}$ is compact in V,

(IV) Ω and $V-\bar{\Omega}$ have the same boundary in V,

(V) each component of $V-\Omega$ is noncompact in V.

We assume moreover that $h\in C^{(1,\lambda)}$.

We shall use the following lemmas as applied to regular solutions in Ω of $\mathfrak{M}u=0$.

LEMMA I (POINCARÉ) [2]. *If w, w_{x_i} are square integrable in a ball B_R of radius R, and if $\bar{w}$ is the average of w over B_R, then*

$$\int_{B_R}(w-\bar{w})^2\,dX \leqq C(B_R)\int_{B_R}\sum_{i=1}^{n}(w_{x_i})^2\,dX,$$

where $C(B_R)$ denotes a constant which depends only on B_R.

LEMMA II (J. MOSER) [3]. *If u is a solution of $\mathfrak{M}u=0$ which is defined in $|x|<2R$ then, for $|x|\leqq R$*

$$u^2(x)\leqq CR^{-n}\int_{|x|<2R}u^2\,dX,$$

where C denotes a constant.

We now give a bound for the first derivatives of a regular solution of $\mathfrak{M}u=0$ in terms of

$$M_\Omega(u)=\int_\Omega a_{ik}\,\partial u/\partial x_i\,\partial u/\partial x_k\,dX.$$

THEOREM I. *Let E be a compact subset of Ω. Then for $x\in E$, u a regular solution of $\mathfrak{M}u=0$, one has*

$$|\,\partial u/\partial x_k|_E\leqq C(E)M_\Omega^{1/2}(u),$$

where $C(E)$ denotes a constant depending only on E.

PROOF. Let $\delta>0$ be defined such that the distance from E to Bd Ω is greater than 4δ. If we denote by $B(x;R)$ the ball of center x and radius R,

$$B(x,4\delta)\subset\Omega\quad\forall x\in E.$$

Let x_0 be a point of E, and let $G(x;y)$ be Green's function for $B(x_0,2\delta)$. Then:

$$u(x)=\int_{\mathrm{Bd}\,B(x_0,2\delta)}u(y)\partial/\partial\nu_y^*G(y;x)\,d_y\sigma,\qquad x\in B(x_0,2\delta),$$

where $\partial/\partial\nu^*$ denotes the conormal derivative. Hence (cf. [4])

$$\partial u/\partial x_k = \int_{\mathrm{Bd}\, B(x_0,2\delta)} u(y)\,\partial/\partial x_k\,\partial/\partial\nu_y^* G(y;x)d_y\sigma, \qquad x \in B(x_0, 2\delta).$$

Let

$$\bar{u}(x_0; 2\delta) = \int_{B(x_0,2\delta)} u(x)\,dx \Big/ \int_{B(x_0,2\delta)} dX,$$

then

$$\partial u/\partial x_k = \int_{\mathrm{Bd}\, B(x_0,2\delta)} (u(y) - \bar{u}(x_0; 2\delta))\,\partial/\partial x_k\partial/\partial\nu_y^* G(y;x)\,d_y\sigma,$$

$$x \in B(x, 2\delta),$$

$$|\partial u/\partial x_k| \leqq C \max_{\mathrm{Bd}\, B(x_0,2\delta)} |u(y) - \bar{u}(x_0; 2\delta)| \int_{\mathrm{Bd}\, B(x_0,2\delta)} d_y\sigma/|x-y|^{n-1},$$

$$x \in B(x_0, 2\delta).$$

Let ω_n be the area of the $n-1$ sphere:

$$|\partial u/\partial x_k| \leqq C\omega_n \cdot 2^{n-1} \max_{\mathrm{Bd}\, B(x_0,2\delta)} |u(x) - \bar{u}(x_0; 2\delta)|, \qquad x \in B(x_0, \delta),$$

$$|\partial u/\partial x_k| \leqq C^{\mathrm{I}} \max_{y \in B(x_0,2\delta)} |u(y) - \bar{u}(x_0; 2\delta)|, \qquad x \in B(x_0, \delta),$$

by the maximum principle. By Lemma II

$$|\partial x/\partial x_k|^2 \leqq C^{\mathrm{II}} \int_{B(x_0,2\delta)} (u(x) - \bar{u}(x_0, 2\delta))^2\,dX \qquad x \in B(x_0, \delta),$$

and by Lemma I

$$\begin{aligned} |\partial u/\partial x_k|^2 &\leqq C^{\mathrm{III}} \int_{B(x_0,4\delta)} \sum_{i=1}^{n} (\partial u/\partial x_i)^2 dX \\ &\leqq C^{\mathrm{III}}\lambda \int_{B(x_0,4\delta)} \lambda^{-1} \sum_{i=1}^{n} (\partial u/\partial x_i)^2\,dX \\ &\leqq C^{\mathrm{IV}} M_{B(x_0,4\delta)}(u), \end{aligned}$$

and C^{IV} depends only on $\mathfrak{M}$ and δ. Cover now E by a finite number, say N, of balls $B(x_j; \delta)$, $j=1, \cdots, n$.

Then

$$|\partial u/\partial x_k|^2 \leqq C^{\mathrm{V}} M_\Omega(u), \qquad x \in E,$$

where $C^{\mathrm{V}} = \max_j C^{\mathrm{IV}}$.

3. Let Ω be regular, and let x_0 be fixed in Ω. Consider a compact set $E \subset \Omega$. Let 4δ be a positive number smaller than the distance from $E \cup \{x_0\}$ to Bd Ω. Cover $E \cup \{x_0\}$ by a finite number of open balls of radius δ, $B(x_0, \delta), \cdots, B(x_N, \delta)$. A point in each $B(x_i; \delta)$ can be joined to x_0 by an arc γ_i in Ω. Let $4\delta'$ be a positive number smaller than the distance from $\bigcup \overline{B}(x_i; \delta) \cup \bigcup \gamma_i$ to Bd Ω and cover each γ_i by a finite number of open balls of radius δ', say $B(y_1, \delta'), \cdots, B(y_s, \delta')$.

Corollary. *Let u be a regular solution of $\mathfrak{M}u = 0$ vanishing at $x = x_0$, then for $x \in E$*

$$|u(x)|^2 \leqq K(E) M_\Omega(u)$$

where $K(E)$ depends only on E (and on x_0).

Proof. It follows from the theorem that if $B(\bar{x}, 4\delta'') \subset \Omega$ where $\operatorname{dist}(\bar{x}, \operatorname{Bd} \Omega) > 4\delta''$ then for x such that $|x - \bar{x}| < \delta$

$$|\operatorname{grad} u|^2 \leqq C(\bar{x}, \delta'') M_{B(\bar{x}, 4\delta'')}(u).$$

Let x', x'' be points in $B(\bar{x}, \delta'')$ then

$$|u(x') - u(x'')| \leqq \int_{x'}^{x''} |\operatorname{grad} u| \, ds \leqq 2\delta'' C^{1/2}(\bar{x}, \delta'') M^{1/2}_{B(\bar{x}, 4\delta'')}(u).$$

Applying the last inequality to the covering defined by $B(x_i, 4\delta)$ and $B(y_j, 4\delta')$ one gets

$$|u(x) - u(x_0)|^2 = |u(x)|^2 \leqq K(E) M_\Omega(u),$$

which proves the corollary.

From the corollary and from the general theory [5] we get immediately the existence of a complete orthonormal system $\{\phi_\nu(x)\}$ and an expansion for Bergman's kernel

$$K(x, y) = \sum_{\nu=1}^{\infty} \phi_\nu(x) \phi_\nu(y),$$

which for fixed x converges uniformly on compact subsets of Ω. The $\phi_\nu(x)$ may be chosen so that $\phi_\nu(x_0) = 0 \ \forall \nu$.

As an application we shall prove the following theorem.

4. Theorem II. *The function $K(x, x)$ is bounded on every compact subset E of Ω.*

Proof. Cf. [6]. From Theorem I we get, for fixed k:

$$|\partial u / \partial x_k|_E \leqq C(E) M_\Omega^{1/2}(u).$$

If u is a solution of $\mathfrak{M}u=0$, regular and such that $\partial u/\partial x_k=1$ at $X_0\in E$, then $M_\Omega(u)\geqq 1/C^2(E)$.

Consider the function

$$\phi^*(x) = \lambda^{-1/2}\sum_{\nu=1}^{N}\partial\phi_\nu/\partial x_k(x_0)\,\phi_\nu(x) \Big/ \sum_{\nu=1}^{N}[\partial\phi_\nu(x_0)/\partial x_k]^2,$$

$$\mathfrak{M}\phi^* = 0 \quad\text{and}\quad \partial\phi^*/\partial x_k(x_0) = 1.$$

Therefore

$$M_\Omega(\phi^*) \leqq \lambda\int_\Omega |\operatorname{grad}\phi^*|^2\,dX = 1 \Big/ \sum_{\nu=1}^{N}[\partial\phi_\nu(x_0)/\partial x_k]^2.$$

Therefore

$$\sum_{\nu=1}^{N}[\partial\phi_\nu/\partial x_k(x_0)]^2 \leqq C^2(E),$$

and

$$\sum_{\nu=1}^{\infty}[\partial\phi_\nu/\partial x_k(x_0)]^2 \leqq C^2(E),$$

and this is true for all $x_0\in E$. An analogous proof works for all k, $k=1,\cdots,n$.

Now, we have

$$\sum_{\nu=1}^{N}[\phi_\nu(x)]^2 = \sum_{\nu=1}^{N}[\phi_\nu(x)-\phi_\nu(x_0)]^2$$

and

$$[\phi_\nu(x)-\phi_\nu(x_0)]^2 \leqq \left[\int_{\gamma(x_0,x)}|\operatorname{grad}\phi_\nu|\,ds\right]^2$$
$$\leqq 4L^2\int_{\gamma(x_0,x)}|\operatorname{grad}\phi_\nu|^2\,ds$$

where $\gamma(x_0,x)$ is an arc from x_0 to x, lying in Ω and of length L; therefore

$$\sum_{\nu=1}^{N}[\phi_\nu(x)]^2 \leqq 4L^2C^2(E),$$

and

$$\sum_{\nu=1}^{\infty} [\phi_\nu(x)]^2 = K(x, x) \leqq 4L^2C^2(E).$$

We are now ready to give the best estimate for $K(E)$ in the corollary. Let p_0 and p_1 be the principal functions for $\mathfrak{M}u=0$ and Ω, defined as in [1]. From Theorem 6 in [1],

$$|u(x)|^2 \leqq M_\Omega(p_0 - p_1)M_\Omega(u)$$

with equality only for $u=a(p_0-p_1)$, $a\in \boldsymbol{R}$. Moreover, Theorem 5 in [1] shows that

$$M_\Omega(u) - 2u(x) = M_\Omega(p_0 - p_1) + M(u - p_0 + p_1),$$

or

$$u(x) = M_\Omega(u, p_0 - p_1).$$

It follows that p_0-p_1, which vanishes at $x=x_0$ is the Bergman kernel for the space of regular solutions of $\mathfrak{M}u=0$ in Ω vanishing at x_0,

$$|u(x)|^2 \leqq K(x, x)M_\Omega(u),$$

and $\sup_E K(x, x)$ is the best possible value for $K(E)$.

Another application of the previous results would be the obtention of the extremal properties of principal functions [1] for open regions V of $\boldsymbol{R}^n$, such that there exists a nested sequence of regular $\{\Omega_n\}$, with the properties $\Omega_{n+1}\supset\bar{\Omega}_n$ and $\bigcup_n \Omega_n = V$.

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