SOME CHARACTERIZATIONS OF SEMI-LOCALLY CONNECTED SPACES

YU-LEE LEE[1]

Let $(X, \mathfrak{U})$ be a connected T_1-space and let $\mathfrak{C}$ be the class of all closed connected subsets of $(X, \mathfrak{U})$. Define the operator K on all the subsets of X as follows: $K(A)$ is the intersection of all the finite unions of elements of $\mathfrak{C}$ which cover A. Then we see that K has the following properties:

(1) If $A \subset X$, then $K(A)$ is closed and $\mathrm{Cl}(A) \subset K(A)$.

(2) If A is connected in $(X, \mathfrak{U})$, then $\mathrm{Cl}(A) = K(A)$.

(3) If A is closed and connected, then $K(A) = A$.

(4) The operator K satisfies the Kuratowski axioms.

Thus K defines a new topology $\mathfrak{V}$ for X, we call $\mathfrak{V}$ the derived topology of $\mathfrak{U}$.

(5) The space $(X, \mathfrak{V})$ is a connected T_1-space.

(6) If A is connected in $(X, \mathfrak{U})$, then A is connected in $(X, \mathfrak{V})$.

(7) If A is closed and connected in $(X, \mathfrak{U})$, then A is closed and connected in $(X, \mathfrak{V})$.

(8) The topology $\mathfrak{V}$ is contained in $\mathfrak{U}$.

Recall from [1] that a connected T_1-space X is said to be semi-locally connected (s.l.c.) at x provided there exists a local open base at x such that $X - V$ has only a finite number of components for any V in the local open base at x. The space X is s.l.c. provided X is s.l.c. at every x in X.

THEOREM 1. *The connected T_1-space* $(X, \mathfrak{U})$ *is s.l.c. if and only if* $\mathrm{Cl}(A) = K(A)$ *for any subsets A of X.*

PROOF. Suppose $(X, \mathfrak{U})$ is s.l.c., $A \subset X$ and $x \notin \mathrm{Cl}(A)$. Then there exists an open neighborhood V of x such that $X - V$ is the union of a finite number of closed connected sets which cover A. Hence $x \notin K(A)$ and therefore $\mathrm{Cl}(A) = K(A)$.

Conversely suppose $\mathrm{Cl}(A) = K(A)$ for all $A \subset X$ and U is an open neighborhood of a point x in X. Since $\mathrm{Cl}(X-U) = K(X-U)$, we have $x \notin K(X-U)$. Therefore there exists a finite family of closed connected sets $C_1, C_2, \cdots, C_n$ which covers $X - U$ and such that $x \notin C_i$ for each $i = 1, 2, \cdots, n$. Since $X - \bigcup \{C_i : i = 1, 2, \cdots, n\}$ is open and is contained in U, it follows that X is s.l.c. at x.

Received by the editors October 7, 1964.

[1] This research was supported by the National Science Foundation under Grant number GP-1457.

THEOREM 2. *Let* $(X, \mathcal{U})$ *be a connected* T_1*-space and let* $\mathcal{V}$ *be the derived topology of* $\mathcal{U}$. *Then* $(X, \mathcal{V})$ *is s.l.c.*

PROOF. Since $(X, \mathcal{V})$ is a connected T_1-space by (5), we can consider the derived topology $\mathcal{W}$ of $\mathcal{V}$ with the defining operator J. Let $\mathcal{C}$ and $\mathcal{C}'$ be the families of all closed connected subsets of $(X, \mathcal{U})$ and $(X, \mathcal{V})$ respectively. By (1), $\mathrm{Cl}(A) \subset K(A) \subset J(A)$ for any $A \subset X$. Since by (7), $\mathcal{C} \subset \mathcal{C}'$, it follows that any finite cover of A by elements of $\mathcal{C}$ is a finite cover of A by elements of $\mathcal{C}'$. Hence by definition of K and J we have $K(A) \supset J(A)$. By Theorem 1, $(X, \mathcal{V})$ is s.l.c.

COROLLARY 1. *A topological space* $(X, \mathcal{V})$ *is s.l.c. if and only if* $\mathcal{V}$ *is the derived topology of some connected* T_1*-space* $(X, \mathcal{U})$.

COROLLARY 2. *A locally connected continuum* $(X, \mathcal{U}))$ *is s.l.c.*

PROOF. Let A be a closed set in $(X, \mathcal{U})$ and $p \notin A$. Then there exists a finite family of closed connected sets which covers A but not p. Hence $p \notin K(A)$ and $K(A) = A$. By Theorem 1, $(X, \mathcal{U})$ is s.l.c.

A mapping f of $(X, \mathcal{U})$ into $(Y, \mathcal{V})$ is called semi-connected if whenever A is a closed connected set in $(Y, \mathcal{V})$, $f^{-1}(A)$ is a closed connected set in $(X, \mathcal{U})$. The following theorem is a generalization of a result of Tanaka [2] and W. J. Pervin and N. Levine [3].

THEOREM 3. *A semi-connected mapping f from* $(X, \mathcal{U})$ *into a s.l.c. space* $(Y, \mathcal{V})$ *is continuous.*

PROOF. Let A be a closed subset of Y; then

$$A = \mathrm{Cl}(A) = K(A)$$
$$= \bigcap \{\bigcup \{C: C \in \mathcal{C}'\}\}$$

where $\mathcal{C}'$ is a finite family of closed connected subsets which covers A. Hence

$$f^{-1}(A) = \bigcap \{\bigcup \{f^{-1}(C): C \in \mathcal{C}'\}\}$$

is closed and f is continuous.

THEOREM 4. *If* $(X, \mathcal{U})$ *is a connected* T_1*-space, then the following statements are equivalent:*

(a) $(X, \mathcal{U})$ *is s.l.c.*

(b) *Every semi-connected mapping f from a topological space Y into* $(X, \mathcal{U})$ *is continuous.*

(c) *The identity mapping from* $(X, \mathcal{V})$ *onto* $(X, \mathcal{U})$ *is continuous where* $\mathcal{V}$ *is the derived topology of* $\mathcal{U}$.

PROOF. The previous theorem shows that (a) implies (b). The identity mapping i is semi-connected, by (7). Hence (b) implies (c). If (c) holds then i is a homeomorphism. By Theorem 2, $(X, \mathfrak{U})$ is s.l.c.

Bibliography

1. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloq. Publ. Vol. 28, Amer. Math. Soc., Providence, R. I., 1942.

2. T. Tanaka, *On the family of connected subsets and the topology of space.*, J. Math. Soc. Japan 7 (1955), 389–393.

3. W. J. Pervin and N. Levine, *Connected mappings of Hausdorff space*, Proc. Amer. Math. Soc. 11 (1960), 688–691.

University of Connecticut and
University of Oregon

A TECHNIQUE FOR CONSTRUCTING EXAMPLES

MARY ELLEN RUDIN

The word *space* in this paper will refer to Hausdorff spaces.

I have recently been asked the following questions.

1 (by the topology class of R. H. Bing). Is there a regular, sequentially compact space in which some nested sequence of continua intersect in a disconnected set?

2 (by E. Michael). Is there a normal, sequentially compact but not compact, space having a separable, metric, locally compact, dense subset?

Examples showing that the answer to both questions is yes, modulo the continuum hypothesis, are easily constructed using a technique I have often used before. The technique, described in §I, is perhaps more interesting than the particular examples which are given in §II. §III gives a variation of the technique and raises some questions.

I. This technique is useful in the construction of pathological spaces having nice dense subsets.

Let R be the wedge in the plane consisting of all points (x, y) such that $0 \leqq x \leqq 1$ and $0 \leqq y \leqq x$; let $T = R - \{(0, 0)\}$.

Let F be the set of all continuous real valued functions whose domain is the set of all positive numbers less than or equal to 1 and whose graph lies in T.

There is a natural partial ordering of the terms of F: if f and g

Received by the editors December 16, 1964.