CELLULARITY AT THE BOUNDARY OF A MANIFOLD

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I. Introduction and definitions. A closed subset $X$ of an $n$-manifold $M^n$ will be said to be cellular at the boundary (CAB) of $M^n$ if there is a sequence $\{B_i^n\}$ of closed $n$-cells in $M^n$ satisfying: $B_i^n \cap [\text{Bd}(M^n)] = B_i^{n-1}$ a closed $(n-1)$-cell, $B_{i+1}^{n-1} \subset \text{Int}(B_i^{n-1})$, $[B_{i+1}^{n-1} \cap \text{Int}(M^n)] \subset \text{Int}(B_i^n \cap \text{Int}(M^n))$, and $\cap_{i=1}^\infty B_i^n = X$. Thus, the notion of CAB is the analogue, for subsets intersecting the boundary of a manifold, of the concept of cellularity introduced by Brown in [1].

Theorem II.2 shows that CAB sets behave like points on the boundary of a manifold. With the aid of a theorem of McMillan's [2], we give criteria for a compact absolute retract to be CAB of a piecewise-linear $n$-manifold for $n \neq 4$. A product theorem for CAB sets is given and with some restrictions on dimensions, we show that subarcs of a CAB arc are either CAB or cellular subsets of the interior of the manifold.

We assume a familiarity with [2], [3], and [4]. $R^n$, $S^n$ denote $n$-space and the $n$-sphere. $D^n(j)$ is the closed $n$-ball in $R^n$ with center at the origin and radius $j$. $I^n(j) = D^{n-1}(j) \times [0, j]$. The empty set is denoted by $\emptyset$.

Let $A, B$ be subsets of an $n$-manifold $M^n$ and let $\delta$ be a positive number. Then $\text{Int}(M^n)$, $\text{Bd}(M^n)$ denote the interior and boundary of $M^n$ respectively, $d(A, B)$, the distance from $A$ to $B$, $\text{Cl}(A)$, the closure of $A$ in $M^n$, and $V_\delta(A)$, the subset of $M^n$ consisting of points $x$ such that $d(x, A) < \delta$.

Let $M^n$ be an $n$-manifold with nonempty boundary and let $X$ be a subset of $M^n$ such that $X \cap \text{Bd}(M^n) \neq \emptyset$. Then $2M^n$ denotes an $n$-manifold with empty boundary obtained by taking two copies $M^n_1, M^n_2$ of $M^n$ and identifying corresponding boundary points. Similarly, if $X_1, X_2$ are the copies of $X$ in $M^n_1, M^n_2$ respectively, then $2X$ is the subset of $2M^n$ consisting of $X_1 \cup X_2$.

II. The pointlike character of CAB sets.

Lemma II.1. If $X$ is CAB of an $n$-manifold $M^n$, a sequence $\{(B_i^n)\}$ of closed $n$-cells may be picked which satisfy (in addition to the necessary

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conditions for $X$ to be CAB of $M^n$) the following: $(B^n_{i-1})'$ is a flat closed $(n-1)$-cell in $\text{Bd}[(B^n_i)' \cap \text{Int}(M^n)] \approx R^{n-1} \times [0, 1)$, and $\text{Bd}[(B^n_i)'] \cap \text{Int}[(B^n_i)']$ is bicollared in $M^n$.

**Proof.** Let $\{B^n_i\}$ satisfy the necessary conditions for $X$ to be CAB of $M^n$. We can pick an $(n-1)$-cell $F_{i-1}^n$ satisfying: $B^n_{i-1} \subset \text{Int}(F_{i-1}^n)$ \(\subset F_{i-1}^n \subset \text{Int}(B^n_i)\) and $\text{Bd}(F_{i-1}^n)$ is bicollared in $B^n_i$. Then there is a homeomorphism $h_i$ of $B^n_i$ onto $I^n(1)$ such that $h_i(F_{i-1}^n) = D^{n-1}(1)$. There is an $\epsilon_i$, $0 < \epsilon_i < \frac{1}{2}$, such that $d[\text{Int}[h_i(B^n_{i+1}) \cup X), \text{Bd}[I^n(1)] - \text{Int}[D^{n-1}(1)]] > \epsilon_i$. Take $(B^n_i)' = h_i^{-1}[I^n(1 - \epsilon_i)]$. Then $\{(B^n_i)\}'$ is the required sequence.

**Theorem 11.2.** Let $X$ be CAB of an $n$-manifold $M^n$ and let $C^n$ be a closed $n$-cell in $M^n$ satisfying $X \subset C^n$, $[\text{Bd}M^n) \subset [\text{Bd}(C^n) \subset \text{Int}[C^n \cap \text{Bd}(M^n)]$. Then there is a map $h$ of $M^n$ onto itself such that $h| C^1(M^n - C^n) = 1$, $h(C^n) = C^n$, $h(X) = p \in \text{Bd}(M^n)$ and $h| M^n - X$ is a homeomorphism of $M^n - X$ onto $M^n - \{p\}$. Thus, $M^n/X \approx M^n$.

**Proof.** Take a sequence $\{B^n_i\}$ assured by Lemma 11.1. We may assume that $B^n_i \subset C^n$ and $[B^n_i \cap \text{Bd}(C^n)] \subset \text{Int}[C^n \cap \text{Bd}(M^n)]$. In the manner of the proof of Theorem 1 of [1], we inductively pick a sequence $\{h_i\}$ of homeomorphisms of $M^n$ onto itself satisfying: $h_i| C^1(M^n - C^n) = 1$, the diameter of $h_i(B^n_i)$ is less than 1, $h_{i+1}| M^n - B^n_i = h_i| M^n - B^n_i$, and the diameter of $h_{i+1}(B^n_{i+1})$ is less than $1/(i+1)$. Then $h = \lim h_i$ is the required map.

**Corollary 11.3.** Let $\{X_i : i = 1, \ldots, k\}$ be a finite collection of disjoint subsets of an $n$-manifold $M^n$ such that each $X_i$ is either cellular in $\text{Int}(M^n)$ or CAB of $M^n$. Then $M^n/\sim X_i$, where $X$ is the decomposition space obtained by identifying $X_i$ to a point $p_i$, $i = 1, \ldots, k$.

### III. CAB criteria for an absolute retract.

**Lemma III.1 ([5, p. 33]).** If $A$ is a closed subspace of a metrizable space $X$ and if both $A$ and $X$ are absolute retracts, then $A$ is a strong deformation retract of $X$.

**Lemma III.2 (Borsuk [6]).** Every locally contractible compact metrizable space of finite dimension is an absolute neighborhood retract.

**Theorem III.3.** Let $X$ and $Y$ be finite dimensional (metric) compact absolute retracts and let $Y$ be a closed subset of $X$. Then $X/Y$ is a compact absolute retract.

**Proof.** Obviously, $X/Y$ is compact and finite dimensional. By Theorem 2.2 [7, p. 123], $X/Y$ is a metric space. By Lemma III.1, $Y$ is a strong deformation retract of $X$. Thus, if $f$ is the quotient map.
of $X$ onto $X/Y$ with $f(Y) = y$, $X/Y$ is contractible to the point $y$. This implies that $X/Y$ is locally contractible and by Lemma III.2, $X/Y$ is an absolute neighborhood retract. Finally, $X/Y$ is a compact absolute retract since it is a compact contractible absolute neighborhood retract.

Theorem III.3 will be applied to situations where $X$ is a compact absolute retract in a manifold $M$ and $Y = X \cap \text{Bd}(M)$ is a compact absolute retract in $\text{Bd}(M)$. Then $X/Y$ is a compact absolute retract in $M/Y$. If $Y$ is CAB of $M$, $M/Y \approx M$ and we may assume that $Y$ is a point in $\text{Bd}(M)$ to simplify arguments. In this case, $2X$ will be a compact absolute retract in $2M$.

**Lemma III.4.** Let $M^{n-1}$ be an $(n-1)$-manifold topologically embedded in the interior of an $n$-manifold $M^n$, $n > 3$. Let $B^{n-1}$ be a closed $(n-1)$-cell in $M^{n-1}$, $p \in \text{Int}(B^{n-1})$, and let $B^{n-1}$ be locally flat in $M^n$ except at $p$. Then $B^{n-1}$ is also locally flat at $p$ provided it has a one-sided local collar at $p$.

**Proof.** Let $B^n$ be a closed $n$-cell in $M^n$ such that $p \in \text{Int}(B^n)$. $B^{n-1}$ has a one-sided local collar at $p$, thus, there is a homeomorphism $h: I^n(1) \to \text{Int}(B^n)$ such that $p = h(0)$, $h[D^n(1)] \subset \text{Int}(B^{n-1})$, and $h[I^n(1) - D^n(1)] \cap M^{n-1} = \emptyset$. Let $S^{n-1}(\frac{1}{2}) = h[Bd(I^n(\frac{1}{2}))]$. Then $S^{n-1}(\frac{1}{2})$ is locally flat in $\text{Int}(B^n) \approx R^n$ except at $p$. Hence, by the corollary in [8], $S^{n-1}(\frac{1}{2})$ is flat in $\text{Int}(B^n)$. This implies that $B^{n-1}$ is locally flat at $p$.

**Lemma III.5.** Let $X$ be a compact subset of an $n$-manifold $M^n$, $n > 3$. Then $X$ is CAB of $M^n \equiv X \cap \text{Bd}(M^n) = Y$ is a cellular subset of $\text{Bd}(M^n)$ and $X$ is cellular in $2M^n$.

**Proof.** The necessity follows from the definition of CAB and the fact that $\text{Bd}(M^n)$ is bicollared in $2M^n$. Thus, suppose $Y$ is a cellular subset of $\text{Bd}(M^n)$ and $X$ is cellular in $2M^n$. We consider $X = X_1$, a subset of $M^n_1$, where $2M^n = M^n_1 \cup M^n_2$ joined along their boundaries. Let $f$ be the quotient map of $2M^n$ onto $2M^n/X$ and let $f(X) = p$. Cellular subsets of the boundary of a manifold are trivially CAB of the manifold since the boundary is collared in the manifold. Therefore, $Y$ is CAB of $M^n_1$ and by Theorem II.2, $f(M^n_2) \approx M^n_2$. Thus, $f[\text{Bd}(M^n_1)]$ is collared in $f(M^n_2)$ and by Lemma III.4, $f[\text{Bd}(M^n_1)]$ is locally flat in $f(2M^n)$ at $f(X) = p$. Hence, we pick a sequence $\{B^n_1\}$ of closed $n$-cells in $f(M^n_1)$ satisfying the conditions necessary for $p$ to be CAB of $f(M^n_1)$ and such that $f^{-1}(B^n_1)$ lies in the interior of some closed $n$-cell in $2M^n$ containing $X$. Then $\{f^{-1}(B^n_1)\}$ is a sequence of closed $n$-cells in $M^n_1$ satisfying the conditions necessary for $X$ to be CAB of $M^n_1$. 
Theorem III.6. Let $X$ be a compact subset of a piecewise-linear $n$-manifold $M^n$, $n > 5$, such that $X$ and $X \cap \partial(M^n) = Y$ are absolute retracts. Then $X$ is CAB of $M^n$ if for each open set $U$ of $M^n$ containing $X$, there is an open set $V$ of $M^n$ such that $X \subset V \subset U$ and: (1) each loop in $V - X$ is homotopic in $U - X$ to a loop in $\partial(M^n)$ and (2) each loop in $(V - X) \cap \partial(M^n)$ is nullhomotopic in $(U - X) \cap \partial(M^n)$.

Proof. The necessity is obvious in view of Lemma III.1. Thus, we show the sufficiency. We will do this by showing that $Y$ is cellular in $\partial(M^n)$, $X$ is cellular in $2M^n$, and applying Lemma III.5. We consider $X = X_1$ a subset of $M^n_1$, where $2M^n = M^n_1 \cup M^n_2$ joined along their boundaries. Condition (2) together with Theorem 1 of [2] imply that $Y$ is cellular in $\partial(M^n_1)$ and hence simultaneously CAB of $M^n_1$ and $M^n_2$. Theorem III.3 shows that $X / Y$ is a compact absolute retract. Thus, we may assume that $Y = y$ is a point in $\partial(M^n)$.

Let $U$ be an open set in $2M^n$ containing $X$. We may assume that $U \cap \partial(M^n_1) = U \cap \partial(M^n_2)$ is an open $(n - 1)$-cell since $Y = y$ is a point. Let $U_1 = U \cap M^n_1$. Then $U_1$ is an open set in $M^n_1$ containing $X$. By hypothesis, there is a set $V_1$ open in $M^n_1$ such that $X \subset V_1 \subset U_1$ and each loop in $V_1 - X$ is homotopic in $U_1 - X$ to a loop in $\partial(M^n_1)$. We may also assume that $V_1 \cap \partial(M^n_1)$ is an open $(n - 1)$-cell whose closure $B^{n-1}_1$ is a closed $(n - 1)$-cell contained in $U_1 \cap \partial(M^n_1)$. There is a positive number $\epsilon$ and a homeomorphism $h : B^{n-1} \times [0, \epsilon) \to U \cap \partial(M^n_1)$ such that $h|B^{n-1} \times \{0\}$ is the inclusion map and $h[B^{n-1} \times (0, \epsilon)] \subset \text{Int}(M^n_1)$. Let $V = V_1 \cup h[\text{Int}(B^{n-1}) \times [0, \epsilon)]$. We will show that any loop in $V - X$ is nullhomotopic in $U - X$.

Let $f : S^1 \to V - X$. We assume that $f$ is simplicial and $f(S^1)$ is in general position with respect to $\partial(M^n_1)$. If $f(S^1) \cap M^n_1 = \emptyset$, the result follows trivially. Thus, suppose $f(S^1) \cap M^n_1 \neq \emptyset$. Then $f(S^1) \cap M^n_1$ consists of a finite number of paths in $V_1 - X$ with endpoints in $\partial(M^n_1)$. Let $\alpha_i$ be one such path with endpoints $p_i, \phi_i$. Then $p_i, \phi_i$ can be joined by an arc $\beta_i$ in $(V_1 - X) \cap \partial(M^n_1)$. If $l_i = \alpha_i \cup \beta_i$, by hypothesis, $l_i$ is homotopic in $U_1 - X$ to a loop in $\partial(M^n_1)$ and hence is nullhomotopic in $U_1 - X$ since $U_1 \cap \partial(M^n_1)$ is an open $(n - 1)$-cell. This implies that $\alpha_i$ is homotopic in $U_1 - X$ to $\beta_i$ with $p_i, \phi_i$ fixed throughout the homotopy. Since $V \cap M^n_2 = h[\text{Int}(B^{n-1}) \times [0, \epsilon)]$, $f(S^1) \cap M^n_2$ is homotopic in $V \cap M^n_2$ to a subset of $\text{Int}(B^{n-1}) - y$ with the homotopy fixed throughout on $\text{Int}(B^{n-1})$. Thus, $f(S^1)$ is homotopic in $U - X$ to a loop in $(U - X) \cap \partial(M^n_1)$ and hence is nullhomotopic in $U - X$. Theorem 1 of [2] implies that $X$ is cellular in $2M^n$ and Lemma III.5 shows that $X$ is CAB of $M^n$.

Remark. Theorem III.6 holds for $n = 5$ if we replace condition (2) by condition (2') requiring $Y$ to be a cellular subset of $\partial(M^n)$. 
Lemma III.7. Let $X$ be a closed subset of $I^n(1)$. Then $X$ is CAB of $I^n(1)$ if and only if $X \subset \partial [I^n(1)] = Y$ is a cellular subset of $\partial [I^n(1)]$ and $2X$ is cellular in $2I^n(1) \approx S^n$.

Proof. The necessity is obvious. Thus, we show the sufficiency. As usual, $2I^n(1) = I^n_1(1) \cup I^n_2(1)$ joined along their boundaries. We may assume that $Y = \gamma$ is a point of $\partial [I^n_1(1)]$ since $Y$ is CAB of $I^n_1(1)$. Let $f : 2I^n(1) \to 2I^n(1)/2X \approx S^n$ be the quotient map with $f(2X) = f(\gamma) = \gamma$. Now $f[\partial [I^n_1(1)]]$ is locally flat in $f[2I^n(1)]$ except possibly at $\gamma$. If $n \neq 3$, $f[\partial [I^n_1(1)]]$ is flat. If $n = 3$, either $f[I^n_1(1)]$ or $f[I^n_2(1)]$ is a closed 3-cell [9]. In either case, we may assume without loss of generality that $f[I^n_1(1)]$ is a closed $n$-cell. The completion of the proof follows as in the proof of Lemma III.5.

Theorem III.8. Let $X$ be a compact subset of a piecewise-linear 3-manifold $M^3$ such that $X$ and $X \subset \partial (M^3)$ are absolute retracts and such that for some open set $0$ of $M^3$ containing $X$, the pair $(0, 0 \cap \partial (M^3))$ is embeddable in $(I^3(1), \partial [I^3(1)])$. Then $X$ is CAB of $M^3$ if for each open set $U$ of $M^3$ containing $X$, there is an open set $V$ of $M^3$ with $X \subset V \subset U$ and each loop in $V - X$ is nullhomotopic in $U - X$.

Proof. The hypothesis on 0 allows us to assume that $M^3 = I^3(1)$. $Y$ is cellular in $\partial [I^3(1)]$ since it is a compact absolute retract in the interior of a 2-manifold. Hence, we assume that $Y = \gamma$ is a point of $\partial [I^3(1)]$.

Let $U$ be an open set of $2I^3(1) = I^3_1(1) \cup I^3_2(1) \approx S^3$ containing $2X = X_1 \cup X_2$. We may assume that $U$ is symmetric with respect to $I^3_1(1)$ and $I^3_2(1)$, and that $U \cap \partial [I^3_1(1)]$ is an open 2-cell. Then by hypothesis and a little care, we obtain an open set $V$ of $2I^3(1)$ such that $V$ is symmetric with respect to $I^3_1(1)$ and $I^3_2(1)$, $V \cap \partial [I^3_1(1)]$ is an open 2-cell, $X_i \subset V_i := [V \cap I^3_i(1)] \subset U_i := [U \cap I^3_i(1)]$, and each loop in $V_i - X_i$ is nullhomotopic in $U_i - X_i$.

Let $f : S^1 \to V - 2X$. We suppose that $f$ is simplicial and that $f(S^1)$ is in general position with respect to $\partial [I^3_1(1)]$. Then $f(S^1) \cap I^3_1(1)$ is a finite collection of paths in $V_i - X$ with endpoints in $\partial [I^3_i(1)]$. As in the proof of Theorem III.6 we join these endpoints with arcs in $(V_i - X) \cap \partial [I^3_i(1)]$ and obtain a homotopy pulling $f(S^1)$ into $(U - X) \cap \partial [I^3_i(1)]$. Since $(U - X) \cap \partial [I^3_1(1)]$ and $(V - X) \cap \partial [I^3_1(1)]$ are open 2-cells, there is another homotopy pulling $f(S^1)$ into $(V_i - X) \cap \partial [I^3_i(1)] = (V_i - X) \cap \partial [I^3_i(1)]$ and then, by hypothesis, it is nullhomotopic in $U_i - X$. Thus, by Theorem 1' of [2], $2X$ is cellular in $2I^3(1)$, and by Lemma III.7, $X$ is CAB of $I^3(1) = M^3(1)$. 
Lemma III.9. Let $X$ be a compact subset of a piecewise-linear 3-manifold $M^3$. Then $X$ is CAB of $M^3 \Rightarrow X \cap \text{Bd}(M^3) = Y$ is cellular in $\text{Bd}(M^3)$ and $2X$ is cellular in $2M^3$.

Proof. As usual, the necessity is obvious. Thus, we show the sufficiency. Let $U$ be an open set of $M^3$ containing $X$. Then $2U = U_1 \cup U_2$ is an open set of $2M^3 = M_1^3 \cup M_2^3$ containing $2X = X_1 \cup X_2$. By hypothesis, there is a closed 3-cell $B^3$ such that $2X \subset \text{Int}(B^3) \subset B^3 \subset U$. By Theorem 3 of [2], we may assume that $B^3$ is a piecewise-linear 3-cell. We also suppose that $\text{Bd}(B^3)$ is in general position with respect to $\text{Bd}(M^3)$. Then $\text{Bd}(B^3) \cap \text{Bd}(M^3)$ consists of a finite number of simple closed curves. We may assume that $B^3$ has been cut down, by removing inessential simple closed curves on $\text{Bd}(B^3)$, to a submanifold $N^3$ such that $2X \subset \text{Int}(N^3)$, $N^3 \cap M^3 = N_i$ is a cube with handles, and $N^3 \cap \text{Bd}(M^3) = D$ is a disk with holes. If $D$ is a disk, we are through. If $D$ is a disk with $n$ holes, we may "cut one of the handles" of either $N_1$ or $N_2$ to reduce $D$ to a disk with $(n - 1)$ holes. By induction, we obtain a closed 3-cell $(B^3)'$ either in $U_1$ or $U_2$ of the required type to show that either $X_i$ is CAB of $M^3$ or $X_2$ is CAB of $M^3$.

Theorem III.10. Let $X$ be a compact 1-dimensional subset of a piecewise-linear 3-manifold $M^3$ such that $X$ and $X \cap \text{Bd}(M^3) = Y$ are absolute retracts. Then $X$ is CAB of $M^3 \iff$ for each open set $U$ of $M^3$ containing $X$, there is an open set $V$ of $M^3$ such that $X \subset V \subset U$ and each loop in $V - X$ is nullhomotopic in $U - X$.

Proof. $2X$ is a compact absolute retract in $2M^3$. D. R. McMillan pointed out to the author that some neighborhood of $2X$ is embeddable in $R^4$ since $2X$ is 1-dimensional. A proof similar to that of Theorem III.8 shows that $2X$ is cellular in $2M^3$. Again, $Y$ is cellular in $\text{Bd}(M^3)$ and thus Lemma III.9 implies that $X$ is CAB of $M^3$.

Theorem III.11. Let $X$ be a compact subset of an i-manifold $M^i$, $i = 1, 2$, such that $X$ and $X \cap \text{Bd}(M^i) = Y$ are absolute retracts. Then $X$ is CAB of $M^i$ and hence $M^i/X \approx M^i$.

Proof. The case $i = 1$ is trivial. If $i = 2$, $Y$ is cellular in $\text{Bd}(M^2)$, $2X$ is cellular in $2M^2$ and an easy argument completes the proof.

IV. CAB sets in products.

Lemma IV.1. Let $N^n$, $M^m$ be $n$, $m$ manifolds respectively such that $\text{Bd}(N^n) \neq \emptyset$ and $\text{Bd}(M^m) = \emptyset$. Then $2(N^n \times M^m) \approx (2N^n) \times M^m$.

Proof. $2(N^n \times M^m)$ consists of two copies of $N^n \times M^m$ joined along
\[ \text{Bd} (N^n \times M^m) = [B(N^n) \times M^m] \cup [N^n \times \text{Bd}(M^m)] \], while \((2N^n) \times M^m\) consists of two copies of \(N^n \times M^m\) joined along \(\text{Bd}(N^n) \times M^m\).

**Theorem IV.2.** Let \(N^n, M^m\) be piecewise-linear \(n, m\) manifolds respectively such that \(\text{Bd}(N^n) \neq \emptyset\), \(\text{Bd}(M^m) = \emptyset\), and \(n \geq 2, m \geq 1\). Let \(X\) be a compact subset of \(N^n, Z\) a compact subset of \(M^m\), and suppose \(X, [X \cap \text{Bd}(N^n)] = Y,\) and \(Z\) are absolute retracts. If \(m + n \geq 6\), then \(X \times Z\) is CAB of \(N^n \times M^m\).

**Proof.** Theorem 8 of [2] implies that \(Y \times Z\) is cellular in \(\text{Bd}(N^n) \times M^m\). It also implies that \(X \times Z\) is cellular in \((2N^n) \times M^m\). Then Lemma IV.1 implies that \(X \times Z\) is cellular in \(2(N^n \times M^m)\). Hence by Lemma III.5, \(X \times Z\) is CAB of \(N^n \times M^m\).

A couple of applications of the corollary to Theorem 8 in [2] together with Lemma III.7 give the following theorem.

**Theorem IV.3.** Let \(X\) be a compact subset of \(D^n(1)\), such that \(X\) and \(X \cap \text{Bd}[D^n(1)]\) are absolute retracts. Then \(X \times X\) is CAB of \(D^n(1) \times [−1, 1]\).

**V. CAB arcs.** Let \(α\) be the arc described in Example 1.3 of [10]. We suppose that \(α \subset I^1(1)\) and \(α \cap \text{Bd}[I^1(1)] = \{ p \}\), where \(p\) is the "good" endpoint of \(α\). Then \(α\) is the monotone union of subarcs each of which is cellular in \(\text{Int}[I^1(1)]\) and each of which contains the "bad" endpoint of \(α\), but \(α\) is not CAB of \(I^1(1)\) since \(2α\) is not cellular in \(2I^1(1)\).

However, going in the other direction we have the following theorem.

**Theorem V.1.** Let \(α\) be an arc CAB of an \(n\)-manifold \(M^n\) and let \(β\) be a subarc of \(α\). Then the following hold:

1. \(β \subset \text{Bd}(M^n), n \neq 5 \Rightarrow \text{β is cellular in } \text{Bd}(M^n)\) and hence CAB of \(M^n\),
2. \(β \cap \text{Bd}(M^n)\) is a point (\(\emptyset\)), \(n \neq 4 \Rightarrow \text{β is CAB of } M^n\) (cellular in \(\text{Int}(M^n)\)),
3. \(β \cap \text{Bd}(M^n)\) is a proper subarc of \(β, n \neq 4, 5 \Rightarrow \text{β is CAB of } M^n\).

**Proof.** Since \(α\) is CAB of \(M^n\), we may assume that \(M^n = I^n(1)\). By Lemma III.7, \(α \cap \text{Bd}[I^n(1)] = σ\) is cellular in \(\text{Bd}[I^n(1)] \approx S^{n−1}\) and \(2α\) is cellular in \(2I^n(1) \approx S^n\). Also \(σ\) is CAB of \(I^n(1), \sigma\) is cellular in \(2I^n(1)\), and hence \(2(α/σ)\) is cellular in \(2(I^n(1)/σ) \approx 2I^n(1) \approx S^n\). Theorem 6 of [2] together with Lemma III.7 give (1) and (2) immediately and (3) follows with an additional easy argument.
References


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