CELLULARITY AT THE BOUNDARY OF A MANIFOLD[1]

CHARLES GREATHOUSE

I. **Introduction and definitions.** A closed subset X of an n-manifold M^n will be said to be *cellular at the boundary* (CAB) of M^n if there is a sequence $\{B_i^n\}$ of closed n-cells in M^n satisfying: $B_i^n \cap [\text{Bd}(M^n)] = B_i^{n-1}$ a closed $(n-1)$-cell, $B_{i+1}^{n-1} \subset \text{Int}(B_i^{n-1})$, $[B_{i+1}^n \cap \text{Int}(M^n)] \subset \text{Int}[B_i^n \cap \text{Int}(M^n)]$, and $\bigcap_{i=1}^{\infty} B_i^n = X$. Thus, the notion of CAB is the analogue, for subsets intersecting the boundary of a manifold, of the concept of cellularity introduced by Brown in [1].

Theorem II.2 shows that CAB sets behave like points on the boundary of a manifold. With the aid of a theorem of McMillan's [2], we give criteria for a compact absolute retract to be CAB of a piecewise-linear n-manifold for $n \neq 4$. A product theorem for CAB sets is given and with some restrictions on dimensions, we show that subarcs of a CAB arc are either CAB or cellular subsets of the interior of the manifold.

We assume a familiarity with [2], [3], and [4]. R^n, S^n denote n-space and the n-sphere. $D^n(j)$ is the closed n-ball in R^n with center at the origin and radius j. $I^n(j) = D^{n-1}(j) \times [0, j]$. The empty set is denoted by $\emptyset$.

Let A, B be subsets of an n-manifold M^n and let δ be a positive number. Then $\text{Int}(M^n)$, $\text{Bd}(M^n)$ denote the interior and boundary of M^n respectively, $d(A, B)$, the distance from A to B, $\text{Cl}(A)$, the closure of A in M^n, and $V_\delta(A)$, the subset of M^n consisting of points x such that $d(x, A) < \delta$.

Let M^n be an n-manifold with nonempty boundary and let X be a subset of M^n such that $X \cap \text{Bd}(M^n) \neq \emptyset$. Then $2M^n$ denotes an n-manifold with empty boundary obtained by taking two copies M_1^n, M_2^n of M^n and identifying corresponding boundary points. Similarly, if X_1, X_2 are the copies of X in M_1^n, M_2^n respectively, then $2X$ is the subset of $2M^n$ consisting of $X_1 \cup X_2$.

II. The pointlike character of CAB sets.

LEMMA II.1. *If X is* CAB *of an n-manifold M^n, a sequence $\{(B_i^n)'\}$ of closed n-cells may be picked which satisfy (in addition to the necessary*

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conditions for X to be CAB *of M^n) the following*: $(B_i^{n-1})'$ *is a flat closed* $(n-1)$*-cell in* $\mathrm{Bd}[(B_i^n)']$, $[(B_i^n)' \cap \mathrm{Int}(M^n)] \approx R^{n-1} \times [0, 1)$, *and* $\mathrm{Bd}[(B_i^n)'] - \mathrm{Int}[(B_i^{n-1})']$ *is bicollared in* M^n.

PROOF. Let $\{B_i^n\}$ satisfy the necessary conditions for X to be CAB of M^n. We can pick an $(n-1)$-cell F_i^{n-1} satisfying: $B_{i+1}^{n-1} \subset \mathrm{Int}(F_i^{n-1}) \subset F_i^{n-1} \subset \mathrm{Int}(B_i^{n-1})$ and $\mathrm{Bd}(F_i^{n-1})$ is bicollared in B_i^{n-1}. Then there is a homeomorphism h_i of B_i^n onto $I^n(1)$ such that $h_i(F_i^{n-1}) = D^{n-1}(1)$. There is an ϵ_i, $0 < \epsilon_i < \frac{1}{2}$, such that $d[h_i(B_{i+1}^{n-1} \cup X), \mathrm{Bd}[I^n(1)] - \mathrm{Int}[D^{n-1}(1)]] > \epsilon_i$. Take $(B_i^n)' = h_i^{-1}[I^n(1-\epsilon_i)]$. Then $\{(B_i^n)'\}$ is the required sequence.

THEOREM II.2. *Let X be* CAB *of an n-manifold M^n and let C^n be a closed n-cell in M^n satisfying* $X \subset C^n$, $[X \cap \mathrm{Bd}(M^n)] = [X \cap \mathrm{Bd}(C^n)] \subset \mathrm{Int}[C^n \cap \mathrm{Bd}(M^n)]$. *Then there is a map h of M^n onto itself such that* $h \mid \mathrm{Cl}(M^n - C^n) = 1$, $h(C^n) = C^n$, $h(X) = p \in \mathrm{Bd}(M^n)$ *and* $h \mid M^n - X$ *is a homeomorphism of* $M^n - X$ *onto* $M^n - \{p\}$. *Thus,* $M^n/X \approx M^n$.

PROOF. Take a sequence $\{B_i^n\}$ assured by Lemma II.1. We may assume that $B_1^n \subset C^n$ and $[B_1^n \cap \mathrm{Bd}(C^n)] \subset \mathrm{Int}[C^n \cap \mathrm{Bd}(M^n)]$. In the manner of the proof of Theorem 1 of [1], we inductively pick a sequence $\{h_i\}$ of homeomorphisms of M^n onto itself satisfying: $h_1 \mid \mathrm{Cl}(M^n - C^n) = 1$, the diameter of $h_1(B_1^n)$ is less than 1, $h_{i+1} \mid M^n - B_i^n = h_i \mid M^n - B_i^n$, and the diameter of $h_{i+1}(B_{i+1}^n)$ is less than $1/i+1$. Then $h = \lim_i h_i$ is the required map.

COROLLARY II.3. *Let* $\{X_i \mid i = 1, \cdots, k\}$ *be a finite collection of disjoint subsets of an n-manifold M^n such that each X_i is either cellular in* $\mathrm{Int}(M^n)$ *or* CAB *of M^n. Then $M^n \approx X$, where X is the decomposition space obtained by identifying X_i to a point p_i,* $i = 1, \cdots, k$.

III. CAB criteria for an absolute retract.

LEMMA III.1 ([5, p. 33]). *If A is a closed subspace of a metrizable space X and if both A and X are absolute retracts, then A is a strong deformation retract of X.*

LEMMA III.2 (BORSUK [6]). *Every locally contractible compact metrizable space of finite dimension is an absolute neighborhood retract.*

THEOREM III.3. *Let X and Y be finite dimensional* (*metric*) *compact absolute retracts and let Y be a closed subset of X. Then X/Y is a compact absolute retract.*

PROOF. Obviously, X/Y is compact and finite dimensional. By Theorem 2.2 [7, p. 123], X/Y is a metric space. By Lemma III.1, Y is a strong deformation retract of X. Thus, if f is the quotient map of X onto X/Y with $f(Y)=y$, X/Y is contractible to the point y. This implies that X/Y is locally contractible and by Lemma III.2, X/Y is an absolute neighborhood retract. Finally, X/Y is a compact absolute retract since it is a compact contractible absolute neighborhood retract.

Theorem III.3 will be applied to situations where X is a compact absolute retract in a manifold M and $Y=X\cap\mathrm{Bd}(M)$ is a compact absolute retract in $\mathrm{Bd}(M)$. Then X/Y is a compact absolute retract in M/Y. If Y is CAB of M, $M/Y\approx M$ and we may assume that Y is a point in $\mathrm{Bd}(M)$ to simplify arguments. In this case, $2X$ will be a compact absolute retract in $2M$.

Lemma III.4. *Let M^{n-1} be an $(n-1)$-manifold topologically embedded in the interior of an n-manifold M^n, $n>3$. Let B^{n-1} be a closed $(n-1)$-cell in M^{n-1}, $p\in\mathrm{Int}(B^{n-1})$, and let B^{n-1} be locally flat in M^n except at p. Then B^{n-1} is also locally flat at p provided it has a one-sided local collar at p.*

Proof. Let B^n be a closed n-cell in M^n such that $p\in\mathrm{Int}(B^n)$. B^{n-1} has a one-sided local collar at p, thus, there is a homeomorphism $h\colon I^n(1)\to\mathrm{Int}(B^n)$ such that $p=h(0)$, $h[D^{n-1}(1)]\subset\mathrm{Int}(B^{n-1})$, and $h[I^n(1)-D^{n-1}(1)]\cap M^{n-1}=\varnothing$. Let $S^{n-1}(\frac{1}{2})=h[\mathrm{Bd}(I^n(\frac{1}{2}))]$. Then $S^{n-1}(\frac{1}{2})$ is locally flat in $\mathrm{Int}(B^n)\approx R^n$ except at p. Hence, by the corollary in [8], $S^{n-1}(\frac{1}{2})$ is flat in $\mathrm{Int}(B^n)$. This implies that B^{n-1} is locally flat at p.

Lemma III.5. *Let X be a compact subset of an n-manifold M^n, $n>3$. Then X is* CAB *of $M^n\Leftrightarrow X\cap\mathrm{Bd}(M^n)=Y$ is a cellular subset of* $\mathrm{Bd}(M^n)$ *and X is cellular in $2M^n$.*

Proof. The necessity follows from the definition of CAB and the fact that $\mathrm{Bd}(M^n)$ is bicollared in $2M^n$. Thus, suppose Y is a cellular subset of $\mathrm{Bd}(M^n)$ and X is cellular in $2M^n$. We consider $X=X_1$ a subset of M_1^n, where $2M^n=M_1^n\cup M_2^n$ joined along their boundaries. Let f be the quotient map of $2M^n$ onto $2M^n/X$ and let $f(X)=p$. Cellular subsets of the boundary of a manifold are trivially CAB of the manifold since the boundary is collared in the manifold. Therefore, Y is CAB of M_2^n and by Theorem II.2, $f(M_2^n)\approx M_2^n$. Thus, $f[\mathrm{Bd}(M_i^n)]$ is collared in $f(M_2^n)$ and by Lemma III.4, $f[\mathrm{Bd}(M_i^n)]$ is locally flat in $f(2M^n)$ at $f(X)=p$. Hence, we pick a sequence $\{B_i^n\}$ of closed n-cells in $f(M_1^n)$ satisfying the conditions necessary for p to be CAB of $f(M_1^n)$ and such that $f^{-1}(B_1^n)$ lies in the interior of some closed n-cell in $2M^n$ containing X. Then $\{f^{-1}(B_i^n)\}$ is a sequence of closed n-cells in M_1^n satisfying the conditions necessary for X to be CAB of M_1^n.

THEOREM III.6. *Let X be a compact subset of a piecewise-linear n-manifold M^n, $n>5$, such that X and $X\cap \mathrm{Bd}(M^n)=Y$ are absolute retracts. Then X is* CAB *of $M^n \Leftrightarrow$ for each open set U of M^n containing X, there is an open set V of M^n such that $X\subset V\subset U$ and*: (1) *each loop in $V-X$ is homotopic in $U-X$ to a loop in* $\mathrm{Bd}(M^n)$ *and* (2) *each loop in* $(V-X)\cap \mathrm{Bd}(M^n)$ *is nullhomotopic in* $(U-X)\cap\mathrm{Bd}(M^n)$.

PROOF. The necessity is obvious in view of Lemma II.1. Thus, we show the sufficiency. We will do this by showing that Y is cellular in $\mathrm{Bd}(M^n)$, X is cellular in $2M^n$, and applying Lemma III.5. We consider $X=X_1$ a subset of M_1^n, where $2M^n=M_1^n\cup M_2^n$ joined along their boundaries. Condition (2) together with Theorem 1 of [2] imply that Y is cellular in $\mathrm{Bd}(M_i^n)$ and hence simultaneously CAB of M_1^n and M_2^n. Theorem III.3 shows that X/Y is a compact absolute retract. Thus, we may assume that $Y=y$ is a point in $\mathrm{Bd}(M_i^n)$.

Let U be an open set in $2M^n$ containing X. We may assume that $U\cap\mathrm{Bd}(M_i^n)$ is an open $(n-1)$-cell since $Y=y$ is a point. Let $U_1=U\cap M_1^n$. Then U_1 is an open set in M_1^n containing X. By hypothesis, there is a set V_1 open in M_1^n such that $X\subset V_1\subset U_1$ and each loop in V_1-X is homotopic in U_1-X to a loop in $\mathrm{Bd}(M_i^n)$. We may also assume that $V_1\cap\mathrm{Bd}(M_i^n)$ is an open $(n-1)$-cell whose closure B^{n-1} is a closed $(n-1)$-cell contained in $U_1\cap\mathrm{Bd}(M_i^n)$. There is a positive number ϵ and a homeomorphism $h: B^{n-1}\times[0,\epsilon)\to U\cap M_2^n$ such that $h|B^{n-1}\times 0$ is the inclusion map and $h[B^{n-1}\times(0,\epsilon)]\subset\mathrm{Int}(M_2^n)$. Let $V=V_1\cup h[\mathrm{Int}(B^{n-1})\times[0,\epsilon)]$. We will show that any loop in $V-X$ is nullhomotopic in $U-X$.

Let $f: S^1\to V-X$. We assume that f is simplicial and $f(S^1)$ is in general position with respect to $\mathrm{Bd}(M_i^n)$. If $f(S^1)\cap M_1^n=\varnothing$, the result follows trivially. Thus, suppose $f(S^1)\cap M_1^n\neq\varnothing$. Then $f(S^1)\cap M_1^n$ consists of a finite number of paths in V_1-X with endpoints in $\mathrm{Bd}(M_i^n)$. Let α_i be one such path with endpoints p_i, ϕ_i. Then p_i, ϕ_i can be joined by an arc β_i in $(V_1-X)\cap\mathrm{Bd}(M_i^n)$. If $l_i=\alpha_i\cup\beta_i$, by hypothesis, l_i is homotopic in U_1-X to a loop in $\mathrm{Bd}(M_i^n)$ and hence is nullhomotopic in U_1-X since $U_1\cap\mathrm{Bd}(M_i^n)$ is an open $(n-1)$-cell. This implies that α_i is homotopic in U_1-X to β_i with p_i, ϕ_i fixed throughout the homotopy. Since $V\cap M_2^n=h[\mathrm{Int}(B^{n-1})\times[0,\epsilon)]$, $f(S^1)\cap M_2^n$ is homotopic in $V\cap M_2^n$ to a subset of $\mathrm{Int}(B^{n-1})-y$ with the homotopy fixed throughout on $\mathrm{Int}(B^{n-1})$. Thus, $f(S^1)$ is homotopic in $U-X$ to a loop in $(U-X)\cap\mathrm{Bd}(M_i^n)$ and hence is nullhomotopic in $U-X$. Theorem 1 of [2] implies that X is cellular in $2M^n$ and Lemma III.5 shows that X is CAB of M^n.

REMARK. Theorem III.6 holds for $n=5$ if we replace condition (2) by condition (2′) requiring Y to be a cellular subset of $\mathrm{Bd}(M^n)$.

LEMMA III.7. *Let X be a closed subset of* $I^n(1)$. *Then X is* CAB *of* $I^n(1) \Leftrightarrow X \cap \mathrm{Bd}[I^n(1)] = Y$ *is a cellular subset of* $\mathrm{Bd}[I^n(1)]$ *and* $2X$ *is cellular in* $2I^n(1) \approx S^n$.

PROOF. The necessity is obvious. Thus, we show the sufficiency. As usual, $2I^n(1) = I_1^n(1) \cup I_2^n(1)$ joined along their boundaries. We may assume that $Y = y$ is a point of $\mathrm{Bd}[I_i^n(1)]$ since Y is CAB of $I_i^n(1)$. Let $f: 2I^n(1) \to 2I^n(1)/2X \approx S^n$ be the quotient map with $f(2X) = f(y) = p$. Now $f[\mathrm{Bd}(I_i^n(1))]$ is locally flat in $f[2I^n(1)]$ except possibly at p. If $n \neq 3$, $f[\mathrm{Bd}(I_i^n(1))]$ is flat. If $n = 3$, either $f[I_1^n(1)]$ or $f[I_2^n(1)]$ is a closed 3-cell [**9**]. In either case, we may assume without loss of generality that $f[I_1^n(1)]$ is a closed n-cell. The completion of the proof follows as in the proof of Lemma III.5.

THEOREM III.8. *Let X be a compact subset of a piecewise-linear* 3-*manifold* M^3 *such that X and* $X \cap \mathrm{Bd}(M^3) = Y$ *are absolute retracts and such that for some open set* 0 *of* M^3 *containing X, the pair* $(0, 0 \cap \mathrm{Bd}(M^3))$ *is embeddable in* $(I^3(1), \mathrm{Bd}[I^3(1)])$. *Then X is* CAB *of* $M^3 \Leftrightarrow$ *for each open set U of* M^3 *containing X, there is an open set V of* M^3 *with* $X \subset V \subset U$ *and each loop in* $V - X$ *is nullhomotopic in* $U - X$.

PROOF. The hypothesis on 0 allows us to assume that $M^3 = I^3(1)$. Y is cellular in $\mathrm{Bd}[I^3(1)]$ since it is a compact absolute retract in the interior of a 2-manifold. Hence, we assume that $Y = y$ is a point of $\mathrm{Bd}[I^3(1)]$.

Let U be an open set of $2I^3(1) = I_1^3(1) \cup I_2^3(1) \approx S^3$ containing $2X = X_1 \cup X_2$. We may assume that U is symmetric with respect to $I_1^3(1)$ and $I_2^3(1)$, and that $U \cap \mathrm{Bd}[I_i^3(1)]$ is an open 2-cell. Then by hypothesis and a little care, we obtain an open set V of $2I^3(1)$ such that V is symmetric with respect to $I_1^3(1)$ and $I_2^3(1)$, $V \cap \mathrm{Bd}[I_i^3(1)]$ is an open 2-cell, $X_i \subset V_i = [V \cap I_i^3(1)] \subset U_i = [U \cap I_i^3(1)]$, and each loop in $V_i - X_i$ is nullhomotopic in $U_i - X_i$.

Let $f: S^1 \to V - 2X$. We suppose that f is simplicial and that $f(S^1)$ is in general position with respect to $\mathrm{Bd}[I_i^3(1)]$. Then $f(S^1) \cap I_i^3(1)$ is a finite collection of paths in $V_i - X$ with endpoints in $\mathrm{Bd}[I_i^3(1)]$. As in the proof of Theorem III.6 we join these endpoints with arcs in $(V_i - X) \cap \mathrm{Bd}[I_i^3(1)]$ and obtain a homotopy pulling $f(S^1)$ into $(U - X) \cap \mathrm{Bd}[I_i^3(1)]$. Since $(U - X) \cap \mathrm{Bd}[I_i^3(1)]$ and $(V - X) \cap \mathrm{Bd}[I_i^3(1)]$ are open 2-cells, there is another homotopy pulling $f(S^1)$ into $(V_1 - X) \cap \mathrm{Bd}[I_i^3(1)] = (V - X) \cap \mathrm{Bd}[I_i^3(1)]$ and then, by hypothesis, it is nullhomotopic in $U_1 - X$. Thus, by Theorem 1′ of [2], $2X$ is cellular in $2I^3(1)$, and by Lemma III.7, X is CAB of $I^3(1) = M^3(1)$.

LEMMA III.9. *Let X be a compact subset of a piecewise-linear 3-manifold M^3. Then X is* CAB *of* $M^3 \Leftrightarrow X \cap \mathrm{Bd}(M^3) = Y$ *is cellular in* $\mathrm{Bd}(M^3)$ *and* $2X$ *is cellular in* $2M^3$.

PROOF. As usual, the necessity is obvious. Thus, we show the sufficiency. Let U be an open set of M^3 containing X. Then $2U = U_1 \cup U_2$ is an open set of $2M^3 = M_1^3 \cup M_2^3$ containing $2X = X_1 \cup X_2$. By hypothesis, there is a closed 3-cell B^3 such that $2X \subset \mathrm{Int}(B^3) \subset B^3 \subset U$. By Theorem 3 of [2], we may assume that B^3 is a piecewise-linear 3-cell. We also suppose that $\mathrm{Bd}(B^3)$ is in general position with respect to $\mathrm{Bd}(M_i^3)$. Then $\mathrm{Bd}(B^3) \cap \mathrm{Bd}(M_i^3)$ consists of a finite number of simple closed curves. We may assume that B^3 has been cut down, by removing inessential simple closed curves on $\mathrm{Bd}(B^3)$, to a submanifold N^3 such that $2X \subset \mathrm{Int}(N^3)$, $N^3 \cap M_i^3 = N_i$ is a cube with handles, and $N^3 \cap \mathrm{Bd}(M_i^3) = D$ is a disk with holes. If D is a disk, we are through. If D is a disk with n holes, we may "cut one of the handles" of either N_1 or N_2 to reduce D to a disk with $(n-1)$ holes. By induction, we obtain a closed 3-cell $(B^3)'$ either in U_1 or U_2 of the required type to show that either X_1 is CAB of M_1^3 or X_2 is CAB of M_2^3.

THEOREM III.10. *Let X be a compact 1-dimensional subset of a piecewise-linear 3-manifold M^3 such that X and $X \cap \mathrm{Bd}(M^3) = Y$ are absolute retracts. Then X is* CAB *of* $M^3 \Leftrightarrow$ *for each open set U of M^3 containing X, there is an open set V of M^3 such that $X \subset V \subset U$ and each loop in $V - X$ is nullhomotopic in $U - X$.*

PROOF. $2X$ is a compact absolute retract in $2M^3$. D. R. McMillan pointed out to the author that some neighborhood of $2X$ is embeddable in R^3 since $2X$ is 1-dimensional. A proof similar to that of Theorem III.8 shows that $2X$ is cellular in $2M^3$. Again, Y is cellular in $\mathrm{Bd}(M^3)$ and thus Lemma III.9 implies that X is CAB of M^3.

THEOREM III.11. *Let X be a compact subset of an i-manifold M^i, $i = 1, 2$, such that X and $X \cap \mathrm{Bd}(M^i) = Y$ are absolute retracts. Then X is* CAB *of M^i and hence $M^i/X \approx M^i$.*

PROOF. The case $i = 1$ is trivial. If $i = 2$, Y is cellular in $\mathrm{Bd}(M^2)$, $2X$ is cellular in $2M^2$ and an easy argument completes the proof.

IV. CAB sets in products.

LEMMA IV.1. *Let N^n, M^m be n, m manifolds respectively such that* $\mathrm{Bd}(N^n) \neq \emptyset$ *and* $\mathrm{Bd}(M^m) = \emptyset$. *Then* $2(N^n \times M^m) \approx (2N^n) \times M^m$.

PROOF. $2(N^n \times M^m)$ consists of two copies of $N^n \times M^m$ joined along

$\text{Bd}(N^n \times M^m) = [B(N^n) \times M^m] \cup [N^n \times \text{Bd}(M^m)]$, while $(2N^n) \times M^m$ consists of two copies of $N^n \times M^m$ joined along $\text{Bd}(N^n) \times M^m$.

THEOREM IV.2. *Let N^n, M^m be piecewise-linear n, m manifolds respectively such that* $\text{Bd}(N^n) \neq \emptyset$, $\text{Bd}(M^m) = \emptyset$, *and* $n \geqq 2$, $m \geqq 1$. *Let X be a compact subset of N^n, Z a compact subset of M^m, and suppose X,* $[X \cap \text{Bd}(N^n)] = Y$, *and Z are absolute retracts. If $m+n \geqq 6$, then $X \times Z$ is* CAB *of $N^n \times M^m$.*

PROOF. Theorem 8 of [2] implies that $Y \times Z$ is cellular in $\text{Bd}(N^n) \times M^m = \text{Bd}(N^n \times M^m)$. It also implies that $X \times Z$ is cellular in $(2N^n) \times M^m$. Then Lemma IV.1 implies that $X \times Z$ is cellular in $2(N^n \times M^m)$. Hence by Lemma III.5, $X \times Z$ is CAB of $N^n \times M^m$.

A couple of applications of the corollary to Theorem 8 in [2] together with Lemma III.7 give the following theorem.

THEOREM IV.3. *Let X be a compact subset of $D^n(1)$, such that X and* $X \cap \text{Bd}[D^n(1)]$ *are absolute retracts. Then $X = X \times 0$ is* CAB *of* $D^n(1) \times [-1, 1]$.

V. CAB **arcs.** Let α be the arc described in Example 1.3 of [**10**]. We suppose that $\alpha \subset I^3(1)$ and $\alpha \cap \text{Bd}[I^3(1)] = \{p\}$, where p is the "good" endpoint of α. Then α is the monotone union of subarcs each of which is cellular in $\text{Int}[I^3(1)]$ and each of which contains the "bad" endpoint of α, but α is not CAB of $I^3(1)$ since 2α is not cellular in $2I^3(1)$.

However, going in the other direction we have the following theorem.

THEOREM V.1. *Let α be an arc* CAB *of an n-manifold M^n and let β be a subarc of α. Then the following hold:*

(1) $\beta \subset \text{Bd}(M^n)$, $n \neq 5 \Rightarrow \beta$ *is cellular in* $\text{Bd}(M^n)$ *and hence* CAB *of M^n,*

(2) $\beta \cap \text{Bd}(M^n)$ *is a point* $(\emptyset)$, $n \neq 4 \Rightarrow \beta$ *is* CAB *of M^n (cellular in* $\text{Int}(M^n)$),

(3) $\beta \cap \text{Bd}(M^n)$ *is a proper subarc of β,* $n \neq 4, 5 \Rightarrow \beta$ *is* CAB *of M^n.*

PROOF. Since α is CAB of M^n, we may assume that $M^n = I^n(1)$. By Lemma III.7, $\alpha \cap \text{Bd}[I^n(1)] = \sigma$ is cellular in $\text{Bd}[I^n(1)] \approx S^{n-1}$ and 2α is cellular in $2I^n(1) \approx S^n$. Also σ is CAB of $I^n(1)$, σ is cellular in $2I^n(1)$, and hence $2(\alpha/\sigma)$ is cellular in $2(I^n(1)/\sigma) \approx 2I^n(1) \approx S^n$. Theorem 6 of [2] together with Lemma III.7 give (1) and (2) immediately and (3) follows with an additional easy argument.

References

1. M. Brown, *A proof of the generalized Schoenflies theorem*, Bull. Amer. Math. Soc. 66 (1960), 74–76.

2. D. R. McMillan, Jr., *A criterion for cellularity in a manifold*, Ann. of Math. 79 (1964), 327–337.

3. J. R. Stallings, *The piecewise-linear structure of euclidean space*, Proc. Cambridge Philos. Soc. 58 (1962), 481–488.

4. M. Brown, *Locally flat imbeddings of topological manifolds*, Ann. of Math. 75 (1962), 331–341.

5. S. T. Hu, *Homotopy theory*, Academic Press, New York, 1959.

6. K. Borsuk, *Über eine Klasse von lokal zusammenhangenden Räumen*, Fund. Math. 19 (1932), 220–240.

7. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloq. Publ. Vol. 28, Amer. Math. Soc., Providence, R. I., 1942.

8. J. C. Cantrell, *Almost locally flat embeddings of S^{n-1} in S^n*, Bull. Amer. Math. Soc. 69 (1963), 716–718.

9. O. G. Harrold, Jr. and E. E. Moise, *Almost locally polyhedral spheres*, Ann. of Math. (2) 57 (1953), 575–578.

10. E. Artin and R. H. Fox, *Some wild cells and spheres in three-dimensional space*, Ann. of Math. 49 (1948), 979–990.

The University of Tennessee and
Vanderbilt University