A NOTE ON THE TWO CARDINAL PROBLEM

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In this note we prove a theorem concerning the two cardinal problem (see [1], [4], [6], [7] for reference and for some of the standard notation); this result has been referred to in [6, p. 311] and [7, (3.7)]. The problem, first proposed by Vaught, is as follows. Let T be a first-order theory and let U be a unary predicate symbol in the language of T. T is said to admit the pair α, β of cardinals if there exists a model $M = \langle A, U, S, \cdots \rangle$ of T such that $|A| = \alpha$ and $|U| = \beta$. Suppose T admits a pair α, β where $\alpha > \beta \geq \omega$. Then what other pairs of cardinals γ, δ must T admit? The following theorem gives a partial answer.

THEOREM. *Let T be a theory in a countable first-order language L with identity. Assume the generalized continuum hypothesis. If T admits a pair α, β where $\alpha > \beta \geq \omega$, then T admits all pairs δ^+, δ where δ is a regular infinite cardinal.*

PROOF. We first note that in the case $\delta = \omega$ the theorem is already known [4, Theorem 6.2], and in fact this particular case can be proved without assuming the generalized continuum hypothesis. Henceforth we assume that δ is a regular infinite cardinal greater than ω.

Let us suppose that L has, in addition to the unary predicate symbol U, some other predicate symbols $S, \cdots$, and, for the sake of definiteness, let us assume that S is a ternary predicate symbol. Let

$$\langle A, U, S, \cdots \rangle$$

be a model of T such that $|A| = \alpha$ and $|U| = \beta$, where $\alpha > \beta \geq \omega$. Let R be a new binary relation over A such that $R \subset U^2$ and R indexes all the finite subsets of U. That is,

(1) for any $u_1, \cdots, u_n \in U$, there exists a $u \in U$ such that for all $t \in U$,

$$R(u, t) \text{ if and only if } t \in \{u_1, \cdots, u_n\}.$$

It is always possible to construct such a relation R as U is infinite.

Let L_1 be the first-order language obtained from L by adding the new binary predicate symbol R. The model

$$\langle A, U, R, S, \cdots \rangle$$

is then a model for L_1. In view of the downward Löwenheim-Skolem theorem as formulated in [5], and since L_1 is still countable, there exists a subset $B \subset A$ such that

(2)
$$U \subset B, \ |B| = \beta, \ \text{and}$$
$$\langle B, U, R, S \cap B^3, \cdots \rangle \prec \langle A, U, R, S, \cdots \rangle.$$

Let L_2 be the first-order language obtained from L_1 by adding the new unary predicate symbol B. Thus

$$\langle A, B, U, R, S, \cdots \rangle$$

is a model for L_2. Let T' be the complete theory given by $\langle A, B, U, R, S, \cdots \rangle$. We note in passing that, by (1), T' contains in particular all closures of formulas of the form

(3)
$$U(z_1) \wedge \cdots \wedge U(z_m) \wedge \phi(z_1) \wedge \cdots \wedge \phi(z_m)$$
$$\rightarrow (\exists y)(R(y, z_1) \wedge \cdots \wedge R(y, z_m) \wedge (\forall t)(R(y, t) \rightarrow \phi(t))),$$

where ϕ is an arbitrary formula of L_2 which may contain other free variables.

Clearly T' is consistent and has an infinite model. It is well known that, assuming the generalized continuum hypothesis, for each regular infinite cardinal $\delta > \omega$, T' has a δ-saturated model (see, for example, [4], [3] for some properties of saturated models)

$$N_0 = \langle A_0, B_0, U_0, R_0, S_0, \cdots \rangle$$

of power δ. By (2), both B_0 and U_0 must be infinite, and since N_0 is δ-saturated, it follows that $|B_0| = |U_0| = \delta$. Furthermore, there are enough sentences in T' to insure that the model (in L_1)

$$N_1 = \langle B_0, U_0, R_0, S_0 \cap (B_0)^3, \cdots \rangle$$

is a proper elementary submodel of the model (in L_1)

$$M_0 = \langle A_0, U_0, R_0, S_0, \cdots \rangle.$$

In particular, we have $N_1 \equiv M_0$. More significantly, since N_0 is δ-saturated, both N_1 and M_0 are also δ-saturated. Since N_1 and M_0 have the same power δ, by the uniqueness of saturated models, we have $N_1 \cong M_0$. From these facts we can draw the following fundamental result:

(4) If $M' = \langle A', U', R', S', \cdots \rangle$ is any model which can be elementarily embedded into M_0 by an isomorphic embedding which maps U' onto U_0, then M' has a proper elementary extension $M'' = \langle A'', U'', R'', S'', \cdots \rangle$ such that

$$U' = U'', \qquad R' = R'', \qquad \text{and} \qquad M'' \cong M_0.$$

We shall define by transfinite induction a sequence of models,

$$M_0 = \langle A_0, U_0, R_0, S_0, \cdots \rangle, \cdots, M_\nu = \langle A_\nu, U_0, R_0, S_\nu, \cdots \rangle, \cdots,$$

$$\text{for every ordinal } \nu < \delta^+,$$

satisfying the following: for every $\nu < \delta^+$ we have

(5) $M_0 \cong M_\lambda$ for every ordinal $\lambda \leq \nu$, and M_κ is a proper elementary submodel of M_λ for all ordinals κ, λ such that $\kappa < \lambda \leq \nu$.

Let μ be an ordinal such that $0 < \mu < \delta^+$. Assume that M_ν has been defined for all $\nu < \mu$ and that (5) holds for all $\nu < \mu$. If $\mu = \nu^+$ for some ν, then by (4) we can easily find a model $M_\mu = \langle A_\mu, U_0, R_0, S_\mu, \cdots \rangle$ such that

$$M_0 \cong M_\mu \text{ and } M_\nu \text{ is a proper elementary submodel of } M_\mu.$$

Hence (5) will hold with μ in the place of ν.

Suppose that μ is a limit ordinal. Let

$$M' = \langle A', U_0, R_0, S', \cdots \rangle = \bigcup_{\lambda < \mu} M_\lambda.$$

Clearly M' is a proper elementary extension of each M_λ, $\lambda < \mu$, and $|A'| = \delta$. We prove that

(6) M' is elementarily embeddable into M_0 in such a way that U_0 is mapped onto U_0.

The proof of (6) will take some time.

Let $a \in (U_0)^\delta$ and $a' \in (A')^\delta$ be enumerations of U_0 and A', respectively. In what follows the letter η shall range over limit ordinals including 0. We shall construct by a second transfinite induction two sequences $b \in (A_0)^\delta$ and $b' \in (A')^\delta$ such that for every $\nu < \delta$,

(7) $$(M_0, b \restriction \nu + 1) \equiv (M', b' \restriction \nu + 1);$$

(8)   if $\nu = \eta + 2k$ with $k < \omega$, then $b'_\nu = a'_{\eta+k}$, and
       if $\nu = \eta + 2k + 1$ with $k < \omega$, then $b_\nu = a_{\eta+k}$.

Let $\lambda < \delta$ and let us suppose that the sequences b and b' have been

defined for each $\nu < \lambda$ and that (7) and (8) hold for each $\nu < \lambda$. From (7), we immediately have

$$(9) \qquad (M_0, b \upharpoonright \lambda) \equiv (M', b' \upharpoonright \lambda).$$

Assume that $\lambda = \eta + 2k$ with $k < \omega$. We define $b_\lambda' = a_{\eta+k}'$. Let C be the set of all formulas $\phi(v_0)$ in the language of $(M', b' \upharpoonright \lambda)$ with at most one free variable v_0 such that

$$(M', b' \upharpoonright \lambda) \vDash \phi(b_\lambda').$$

C is finitely satisfiable in $(M', b' \upharpoonright \lambda)$. Hence, by (9), C is finitely satisfiable in $(M_0, b \upharpoonright \lambda)$. Since M_0 is δ-saturated and $|C| < \delta$, we have C is simultaneously satisfiable in $(M_0, b \upharpoonright \lambda)$ by some element $e \in A_0$. We define $b_\lambda = e$. It follows that

$$((M_0, b \upharpoonright \lambda), b_\lambda) \equiv ((M', b' \upharpoonright \lambda), b_\lambda')$$

and

$$(M_0, b \upharpoonright \lambda + 1) \equiv (M', b' \upharpoonright \lambda + 1).$$

Hence (7) and (8) hold with λ in place of ν.

Assume that $\lambda = \eta + 2k + 1$ with $k < \omega$. We define $b_\lambda = a_{\eta+k}$. Let D be the set of all formulas $\phi(v_0)$ in the language of $(M_0, b \upharpoonright \lambda)$ (the same language as that of $(M', b' \upharpoonright \lambda)$) with at most one free variable v_0 such that

$$(M_0, b \upharpoonright \lambda) \vDash \phi(b_\lambda).$$

Notice that the formula $U(v_0) \in D$. D is finitely satisfiable in $(M_0, b \upharpoonright \lambda)$, so by (9), D is finitely satisfiable in $(M', b' \upharpoonright \lambda)$. We shall prove that

(10) D is simultaneously satisfiable in $(M', b' \upharpoonright \lambda)$ by some element $e \in U_0$.

Let $S_\omega(D)$ denote the set of all finite subsets of D. Notice that $|S_\omega(D)| < \delta$. Consider a set $d \in S_\omega(D)$. Let $\phi_d(v_0)$ be the conjunction of all the formulas in $d \cup \{U(v_0)\}$. It follows that

$$(M', b' \upharpoonright \lambda) \vDash (\exists v_0) \phi_d(v_0).$$

Remember that ϕ_d is a formula in the language of $(M', b' \upharpoonright \lambda)$, hence there are a finite number of the terms of the sequence $b' \upharpoonright \lambda$, say $b_{r_1}, \cdots, b_{r_n}$, such that

$$M' \vDash (\exists v_0) \phi_d(v_0, b_{r_1}, \cdots, b_{r_n}),$$

where $\phi_d(v_0, v_1, \cdots, v_n)$ is some formula of L_1. Let $\nu_d < \mu$ be an ordinal such that

$$b_{r_1}, \cdots, b_{r_n} \in A_{r_d}.$$

Then since $M_{r_d} \prec M'$, we have

$$M_{r_d} \vDash (\exists v_0)\phi_d(v_0, b_{r_1}, \cdots, b_{r_n}).$$

We can now select, for each $d \in S_\omega(D)$, the elements $b_{r_1}, \cdots, b_{r_n}$, the ordinal ν_d, and an element $e_d \in U_0$ such that

$$(11) \qquad\qquad M_{r_d} \vDash \phi_d(e_d, b_{r_1}, \cdots, b_{r_n}).$$

Notice that if $d \subset d' \in S_\omega(D)$, then

$$\vdash \phi_{d'}(v_0) \rightarrow \phi_d(v_0)$$

and since $e_{d'} \in U_0 \subset A_{r_d}$,

$$M_{r_d} \vDash \phi_d(e_{d'}, b_{r_1}, \cdots, b_{r_n}).$$

Again let $d \in S_\omega(D)$ and consider the set of conditions

$$E_d = \{ R(v_0, e_{d'}) : d \subset d' \in S_\omega(D) \}$$
$$\cup \{ \forall t(R(v_0, t) \rightarrow \phi_d(t, b_{r_1}, \cdots, b_{r_n})) \}.$$

Using the fact that M_{r_d} is a model of T', by (3), we see that every finite subset of E_d is satisfiable in M_{r_d}. Since M_{r_d} is δ-saturated and $|E_d| < \delta$, we have that E_d is simultaneously satisfiable in M_{r_d}. This means we can select for each $d \in S_\omega(D)$ an element $f_d \in U_0$ such that

$$(12) \qquad\qquad M_{r_d} \vDash \forall t(R(f_d, t) \rightarrow \phi_d(t, b_{r_1}, \cdots, b_{r_n})),$$

and

$$(13) \qquad M_{r_d} \vDash R(f_d, e_{d'}) \quad \text{for all } d' \text{ such that } d \subset d' \in S_\omega(D).$$

Consider now a third set of conditions,

$$F = \{ R(f_d, v_0) : d \in S_\omega(D) \}.$$

If $R(f_{d_1}, v_0), \cdots, R(f_{d_m}, v_0) \in F$, then letting $e' = e_{(d_1 \cup \cdots \cup d_m)}$, we have by (13),

$$M_{r_{d_i}} \vDash R(f_{d_i}, e') \qquad \text{for all } i \text{ such that } 1 \leq i \leq m.$$

Since all the elements involved belong to U_0 and hence to A_0, we have

$$M_0 \vDash R(f_{d_1}, e') \land \cdots \land R(f_{d_m}, e').$$

So F is finitely satisfiable in M_0. Since M_0 is δ-saturated and $|F| < \delta$, we have

$(14) \qquad F$ is simultaneously satisfiable in M_0 by some element

$$e \in U_0.$$

It remains to prove (10). It is sufficient to show that

$$(M', b' \upharpoonright \lambda) \vDash \phi_d(e) \qquad \text{for all } d \in S_\omega(D).$$

So, let $d \in S_\omega(D)$. By (14) we have

$$M_0 \vDash R(f_d, e),$$

and, since $M_0 \prec M_{\nu_d}$,

$$M_{\nu_d} \vDash R(f_d, e).$$

By (12), we have

$$M_{\nu_d} \vDash \phi_d(e, b_{\nu_1}, \cdots, b_{\nu_n}),$$

and, since $M_{\nu_d} \prec M'$,

$$M' \vDash \phi_d(e, b_{\nu_1}, \cdots, b_{\nu_n}).$$

This means $(M', b' \upharpoonright \lambda) \vDash \phi_d(e)$, and (10) is proved. We define $b'_\lambda = e$. It is now immediate that conditions (7) and (8) again hold with λ in place of ν.

This completes the second transfinite induction and (7) and (8) hold for all $\nu < \delta$. It follows that

$$(M_0, b) \equiv (M', b'),$$
$$\text{the range of } b' \text{ is } A', \text{ and}$$
$$\text{the range of } b \text{ includes } U_0.$$

Let h be the mapping

$$h: b'_\nu \to b_\nu, \qquad \text{for all } \nu < \delta.$$

It is a simple matter to verify that h is a one-to-one elementary embedding of M' into M_0 such that U_0 is mapped onto U_0. So (6) has been proved. Using (6) and (4), we can again find a model $M_\mu = \langle A_\mu, U_0, R_0, S_\mu, \cdots \rangle$ such that (5) holds with μ in place of ν. This completes our original transfinite induction and (5) now holds for every $\nu < \delta^+$.

Let $M = \bigcup_{\nu < \delta^+} M_\nu$. It is clear that M is a model of T' of power δ^+ with the set U_0 of power δ. Throwing away the superfluous relation R_0, the resulting model is a model of T. Hence T admits δ^+, δ and the theorem is proved.

REMARKS. (1) The restriction to countable theories T is not necessary. We can establish, again assuming the generalized continuum hypothesis, that: If T has $\kappa \geq \omega$ symbols and T admits $\alpha > \beta \geq \omega$, then T admits every pair of cardinals δ^+, δ where δ is a regular infinite

cardinal greater than κ. This result can be proved on either one of the following two ways. One can use the fact that there exist δ-saturated models of power δ in any theory T with κ symbols and with an infinite model, provided that δ is regular infinite and greater than κ. The argument then parallels the argument given in the theorem. One can also use the fact that, using the theorem, every countable subtheory of T will admit the pair δ^+, δ for a regular δ. If $\delta > \kappa$, then by an easy ultraproduct argument (see [1], for instance) we can show that T admits the pair δ^+, δ. Curiously, neither one of the above arguments will prove the very first case, namely, T admits κ^+, κ if κ is itself regular. Recently, by using the fact that the proof of our theorem involves saturated models, Vaught has shown (private communication, unpublished) that the missing case can be proved. One intriguing problem of the same type remains open. Suppose T has κ^{++} symbols and T admits κ^+, κ. Then must T admit κ^{++}, κ^+?

(2) As the reader can see, the proof of the theorem involves the construction of a tower of models

$$M_0, \cdots, M_\nu, \cdots \qquad \text{for } \nu < \delta^+,$$

with the fixed set U_0. This is the basic idea used by Vaught in his first proof for the case $\delta = \omega$ [4]. The whole trick, and indeed the only trick, of our proof is to show that the process can be continued at the limit ordinals. The introduction of the binary relation R which indexes the finite sets of U is the key to the argument. The trick seems to break down if δ is a singular cardinal and M_0 is taken to be a special model of power δ (in the sense of [4]). Hence, it is still not known if T must admit all pairs δ^+, δ if T admits some pair α, β where $\alpha > \beta \geqq \omega$.

(3) The question also remains open whether this trick, or some similar device, will enable us to show that if T admits α, β where $\alpha > \beta^+$ and $\beta \geqq \omega$, then T must admit all pairs δ^{++}, δ where δ is an infinite regular cardinal. By a simple observation this last result, if provable, surely must require the generalized continuum hypothesis. This is because it follows from the work of Cohen [2] that it is consistent to assume that $2^\omega = \omega_2$ and, say, $2^{\omega_2} = \omega_3$. Hence, one can easily construct a theory T which admits ω_2, ω, but which does not admit ω_4, ω_2. This observation does not seem to show that the generalized continuum hypothesis is necessary for our theorem.

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A REMARK ON WIENER'S TAUBERIAN THEOREM

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A recent note by Levinson [1] made it seem worthwhile to point out that a weaker version of the Tauberian theorem can be proved in a few lines which is, however, strong enough to provide a proof of the prime number theorem.

Let $K(x) \in L(-\infty, \infty)$ and assume that its Fourier transform obeys the standard condition

(1)
$$\kappa(\xi) = \int_{-\infty}^{\infty} K(x)e^{i\xi x}\, dx$$

$$\neq 0 \quad \text{for all } -\infty < \xi < \infty.$$

One version of Wiener's Tauberian theorem is the assertion that if $m(y)$ is a bounded measurable function such that for almost all x,

(2)
$$\int_{-\infty}^{\infty} K(x - y)m(y)\, dy = 0$$

then $m(y) = 0$ almost everywhere.

The weaker version of the Tauberian theorem is obtained by adding an extra requirement on the function $K(x)$, namely that

(3)
$$x^2 K(x) \in L(-\infty, \infty).$$

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