THE RAPIDITY OF CONVERGENCE OF THE HERMITE-FEJÉR APPROXIMATION TO FUNCTIONS OF ONE OR SEVERAL VARIABLES

O. SHISHA AND B. MOND

1. The Hermite-Fejér polynomials $H_n(f, x)$ are important approximating polynomials to a given real function $f(x)$ defined on $[-1, 1]$. For every positive integer n,

$$H_n(f, x) \equiv \sum_{k=1}^{n} f(x_k^{(n)}) A_k^{(n)}(x) \tag{1}$$

where

$$A_k^{(n)}(x) \equiv (1 - x x_k^{(n)})[T_n(x)/\{n(x - x_k^{(n)})\}]^2 \qquad (k = 1, 2, \cdots, n),$$

$$T_n(x) \equiv 2^{n-1} \prod_{k=1}^{n} (x - x_k^{(n)}), \qquad x_k^{(n)} = \cos\left(\frac{2k-1}{2n}\pi\right)$$
$$(k = 1, 2, \cdots, n).$$

For $n = 1, 2, \cdots, T_n(x)$ (the nth degree Tchebycheff polynomial of the first kind) satisfies $T_n(\cos\theta) \equiv \cos(n\theta)$, and we also have

$$A_k^{(n)}(x) \geqq 0 \quad \text{for } k = 1, 2, \cdots, n \text{ and every } x \in [-1, 1], \tag{2}$$

$$\sum_{k=1}^{n} A_k^{(n)}(x) \equiv 1, \tag{3}$$

$$H_n(f, x_k^{(n)}) = f(x_k^{(n)}) \qquad (k = 1, 2, \cdots, n), \tag{4}$$

$$H_n'(f, x_k^{(n)}) = 0 \qquad (k = 1, 2, \cdots, n). \tag{5}$$

2. Suppose f is a real function, continuous in $[-1, 1]$. Then a classical result of Fejér [1] states that $H_n(f, x)$ converges uniformly to $f(x)$ on $[-1, 1]$. As to the rapidity of convergence, E. Moldovan published in [2] the estimate $\max_{-1 \leq x \leq 1} |f(x) - H_n(f, x)| \leqq 2\pi\omega(n^{-1} \log n)$, where ω is the modulus of continuity of f in $[-1, 1]$.

3. One of our purposes in this paper is to construct an analog of the polynomial (1) for functions f of several variables and to study the corresponding rapidity of convergence. Since the Romanian paper

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[2] is inaccessible to many readers and is also wanting, we shall prove in Theorem 2 a result which is essentially the same as that published by Moldovan. Theorems 1 and 2 will be used to obtain Theorems 3 and 4 concerning functions of several variables.

4. THEOREM 1. *Let f be a real function satisfying throughout* $[-1, 1]$

$$|f(v) - f(u)| \leqq \lambda |v - u|$$

where λ is a positive constant. Then for $n = 1, 2, \cdots$ and every $x \in [-1, 1]$,

$$|f(x) - H_n(f, x)| < 4\lambda\pi n^{-1}(\alpha + \log n), \tag{6}$$

where $\alpha = \frac{1}{2} + C - \log 2 = 0.384 \cdots$, C being Euler's constant.

PROOF. Let n be a positive integer and let $-1 \leqq x \leqq 1$. We shall prove (6). Let $x = \cos\theta$ $(0 \leqq \theta \leqq \pi)$, and let $\theta_k = ((2k-1)/2n)\pi$ $(k = 1, 2, \cdots, n)$. Since by (4), $H_n(f, \cos\theta_k) = f(\cos\theta_k)$ for $k = 1, 2, \cdots, n$, we may assume $\theta \neq \theta_k$, $k = 1, 2, \cdots, n$. By (3), (1), and (2)

$$
\begin{aligned}
&|f(x) - H_n(f, x)| \\
&= \left| \sum_{k=1}^{n} [f(x) - f(x_k^{(n)})] A_k^{(n)}(x) \right| \leqq \sum_{k=1}^{n} |f(x) - f(x_k^{(n)})| A_k^{(n)}(x) \\
&= \sum_{k=1}^{n} |f(x) - f(x_k^{(n)})| (1 - x x_k^{(n)}) [T_n(x)/\{n(x - x_k^{(n)})\}]^2 \\
&\leqq \lambda \sum_{k=1}^{n} |x - x_k^{(n)}| (1 - x x_k^{(n)}) [T_n(x)/\{n(x - x_k^{(n)})\}]^2 \\
&= \lambda n^{-2} \sum_{k=1}^{n} |\cos\theta - \cos\theta_k| (1 - \cos\theta\cos\theta_k)[\cos(n\theta)/(\cos\theta - \cos\theta_k)]^2 \\
&< \lambda n^{-2} \sum_{k=1}^{n} |\theta - \theta_k| (1 - \cos\theta\cos\theta_k)[\cos(n\theta)/(\cos\theta - \cos\theta_k)]^2 \\
&\leqq \lambda n^{-2} \sum_{k=1}^{n} |\theta - \theta_k| [1 - \cos\theta\cos\theta_k + \sin\theta\sin\theta_k] \\
&\quad \cdot [(\cos(n\theta) - \cos(n\theta_k))/(\cos\theta - \cos\theta_k)]^2 \\
&= \lambda n^{-2} \sum_{k=1}^{n} |\theta - \theta_k| \, 2\sin^2\{(\theta + \theta_k)/2\} \sin^2\{n(\theta + \theta_k)/2\} \\
&\quad \cdot \sin^2\{n(\theta - \theta_k)/2\} \sin^{-2}\{(\theta + \theta_k)/2\} \sin^{-2}\{(\theta - \theta_k)/2\} \\
&< 2\lambda n^{-2} \sum_{k=1}^{n} |\theta - \theta_k| [\sin\{n(\theta - \theta_k)/2\}/\sin\{(\theta - \theta_k)/2\}]^2.
\end{aligned}
$$

Suppose $n \geqq 4$ and $\theta_j < \theta < \theta_{j+1}$, $2 \leqq j \leqq n-2$. Since $|\sin(ny)/\sin y| < n$ for every real $y \neq 0, \pm\pi, \pm 2\pi, \cdots$, therefore

$$\begin{aligned} &|\theta - \theta_j| \, [\sin\{n(\theta - \theta_j)/2\}/\sin\{(\theta - \theta_j)/2\}]^2 \\ &\qquad + |\theta - \theta_{j+1}| \, [\sin\{n(\theta - \theta_{j+1})/2\}/\sin\{(\theta - \theta_{j+1})/2\}]^2 \\ &\qquad < |\theta - \theta_j| \, n^2 + |\theta - \theta_{j+1}| \, n^2 = n\pi. \end{aligned} \tag{7}$$

Since $y/\sin(y/2)$ is strictly increasing in $(0, \pi)$, we have $y/\sin(y/2) < \pi/\sin(\pi/2) = \pi$ whenever $0 < y < \pi$. For such y one has

$$y[\sin(ny/2)/\sin(y/2)]^2 = y^{-1}[y/\sin(y/2)]^2 \sin^2(ny/2) < \pi^2/y.$$

For $k = 1, 2, \cdots, j-1$, $\theta - \theta_k > (j-k)\pi/n$, and so

$$(\theta - \theta_k)[\sin\{n(\theta - \theta_k)/2\}/\sin\{(\theta - \theta_k)/2\}]^2 < \pi^2/(\theta - \theta_k) < n\pi/(j-k).$$

Consequently,

$$\begin{aligned} &\sum_{k=1}^{j-1} |\theta - \theta_k| \, [\sin\{n(\theta - \theta_k)/2\}/\sin\{(\theta - \theta_k)/2\}]^2 \\ &\qquad < n\pi \sum_{k=1}^{j-1} (j-k)^{-1} = n\pi \sum_{k=1}^{j-1} k^{-1}. \end{aligned} \tag{8}$$

Similarly,

$$\begin{aligned} &\sum_{k=j+2}^{n} |\theta - \theta_k| \, [\sin\{n(\theta - \theta_k)/2\}/\sin\{(\theta - \theta_k)/2\}]^2 \\ &\qquad < n\pi \sum_{k=j+2}^{n} [k - (j+1)]^{-1} = n\pi \sum_{k=1}^{n-1-j} k^{-1}. \end{aligned} \tag{9}$$

From (8), (7) and (9) we obtain

$$\begin{aligned} &\sum_{k=1}^{n} |\theta - \theta_k| \, [\sin\{n(\theta - \theta_k)/2\}/\sin\{(\theta - \theta_k)/2\}]^2 \\ &\qquad < n\pi\left(1 + \sum_{k=1}^{j-1} k^{-1} + \sum_{k=1}^{n-1-j} k^{-1}\right), \end{aligned}$$

and therefore

$$|f(x) - H_n(f, x)| < 2\lambda\pi n^{-1}\left(1 + \sum_{k=1}^{j-1} k^{-1} + \sum_{k=1}^{n-1-j} k^{-1}\right). \tag{10}$$

If n is even, then

$$\sum_{k=1}^{j-1} k^{-1} + \sum_{k=1}^{n-1-j} k^{-1} \leqq \max_{1\leqq q\leqq n-3} \left(\sum_{k=1}^{q} k^{-1} + \sum_{k=1}^{n-2-q} k^{-1} \right)$$
$$= 2[1 + 2^{-1} + \cdots + \{(n-2)/2\}^{-1}].$$

If n is odd, then

$$\sum_{k=1}^{j-1} k^{-1} + \sum_{k=1}^{n-1-j} k^{-1} \leqq \max_{1\leqq q\leqq n-3} \left(\sum_{k=1}^{q} k^{-1} + \sum_{k=1}^{n-2-q} k^{-1} \right)$$
$$= 2[1+2^{-1} + \cdots + \{(n-3)/2\}^{-1}] + \{(n-1)/2\}^{-1}.$$

Thus

$$\begin{aligned} &|f(x) - H_n(f, x)| < 2\lambda\pi n^{-1}(1+2[1+2^{-1}+ \cdots + \{(n-2)/2\}^{-1}]) \\ &\qquad\qquad \text{if } n \text{ is even}, \\ &|f(x) - H_n(f, x)| < 2\lambda\pi n^{-1}(1+2[1+2^{-1}+ \cdots + \{(n-3)/2\}^{-1} \\ &\qquad\qquad +(n-1)^{-1}]) \quad \text{if } n \text{ is odd}. \end{aligned} \tag{11}$$

One can show similarly, that (11) (with $n \geqq 4$) is true for every other position of θ in $[0, \pi]$.

Since $(\sum_{k=1}^{q-1} k^{-1}) - \log q < C$ for $q = 2, 3, 4, \cdots$, the desired inequality (6) follows from (11) for n even and $\geqq 4$.

Consider the sequence

$$a_q \equiv \left(\sum_{k=1}^{q-1} k^{-1} \right) + (2q)^{-1} - \log[(2q+1)/2] \qquad (q = 2, 3, 4, \cdots).$$

For every $q \geqq 2$ we have

$$a_{q+1} - a_q = \tfrac{1}{2}[q^{-1} + (q+1)^{-1}] - \log[1 + 2/(2q+1)]$$
$$> \tfrac{1}{2}[q^{-1} + (q+1)^{-1}] - 2/(2q+1) > 0.$$

Since $\lim_{q\to\infty} a_q = C$, we have $a_q < C$ $(q = 2, 3, 4, \cdots)$.

Suppose n is odd and >4. Setting $q = (n-1)/2$ we have $C > a_q = 1 + 2^{-1} + \cdots + \{(n-3)/2\}^{-1} + (n-1)^{-1} - \log(n/2)$, and (6) follows from (11).

Finally one can verify (6) for $n = 1, 2, 3$ by the same sort of calculations which led to (10).

5. Theorem 2. *Let f be a real function, defined and bounded on $[-1, 1]$. For every $\delta \in [0, 2]$ let*

$$\omega(\delta) = \sup_{-1\leqq u\leqq v\leqq 1;\, v-u\leqq\delta} |f(v) - f(u)|.$$

Then for $n=2, 3, 4, \cdots$ *we have*

$$\sup_{-1\leqq x\leqq 1} |f(x) - H_n(f, x)| \leqq [2 + 4\pi + \epsilon_n]\omega(n^{-1}\log n)$$

where ϵ_n *depends on n only and* $\epsilon_n \to 0$ *as* $n\to\infty$.

PROOF. Let n be a positive integer. Let $x_0, x_1, \cdots, x_N$ ($N\geqq 1$) be reals with $-1=x_0<x_1<\cdots<x_N=1$. Let f_n be the function with domain $[-1, 1]$ such that $f_n(x_k)=f(x_k)$, $k=0, 1, \cdots, N$ and such that f_n is linear in each $[x_{k-1}, x_k]$ ($k=1, 2, \cdots, N$). If $x_{k-1}\leqq u<v\leqq x_k$ for some k, then

$$\begin{aligned}|f_n(v) - f_n(u)|/(v-u) &= |f(x_k) - f(x_{k-1})|/(x_k - x_{k-1})\\ &\leqq \omega(x_k - x_{k-1})/(x_k - x_{k-1}) \leqq \lambda\end{aligned}$$

where

$$\lambda = \max_{1\leqq k\leqq N} [\omega(x_k - x_{k-1})/(x_k - x_{k-1})].$$

Therefore $|f_n(v) - f_n(u)| \leqq \lambda(v-u)$ whenever $-1\leqq u<v\leqq 1$. By Theorem 1 we have throughout $[-1, 1]$,

$$|f_n(x) - H_n(f_n, x)| \leqq 4\lambda\pi n^{-1}(\alpha + \log n). \tag{12}$$

One easily verifies that throughout $[-1, 1]$,

$$|f(x) - f_n(x)| \leqq \mu \tag{13}$$

where $\mu = \max_{1\leqq k\leqq N}\omega(x_k - x_{k-1})$. Therefore, by (1), (2) and (3) we have throughout $[-1, 1]$,

$$|H_n(f, x) - H_n(f_n, x)| \leqq \sum_{k=1}^{n} |f(x_k^{(n)}) - f_n(x_k^{(n)})|\, A_k^{(n)}(x) \leqq \mu. \tag{14}$$

From (12), (13) and (14) we get

$$\sup_{-1\leqq x\leqq 1} |f(x) - H_n(f, x)| \leqq 2\mu + 4\lambda\pi n^{-1}(\alpha + \log n) \tag{15}$$

and so

$$\begin{aligned}&\sup_{-1\leqq x\leqq 1} |f(x) - H_n(f, x)|\\ &\leqq \inf_{-1=x_0<x_1\cdots<x_N=1;\,N=1,2,3,\cdots}\Big\{2\max_{1\leqq k\leqq N}\omega(x_k - x_{k-1})\\ &\qquad + 4\max_{1\leqq k\leqq N}[\omega(x_k - x_{k-1})/(x_k - x_{k-1})]\pi n^{-1}(\alpha + \log n)\Big\}.\end{aligned}$$

If we take in particular for some positive integer N, $x_k=-1+(2k/N)$ $(k=0, 1, \cdots, N)$, then from (15) we obtain

$$\sup_{-1\leqq x\leqq 1} |f(x)-H_n(f,x)| \leqq 2\omega(2/N)[1+\pi N n^{-1}(\alpha+\log n)]. \tag{16}$$

Thus

$$\sup_{-1\leqq x\leqq 1} |f(x)-H_n(f,x)| \leqq \inf_{N=1,2,\cdots} \{2\omega(2/N)[1+\pi N n^{-1}(\alpha+\log n)]\}.$$

Suppose $n\geqq 2$, and let N be the integral part of $1+(2n/\log n)$. Then (16) yields

$$\sup_{-1\leqq x\leqq 1} |f(x)-H_n(f,x)| \leqq [2+4\pi+\epsilon_n]\omega(n^{-1}\log n), \tag{17}$$

where $\epsilon_n=2\pi n^{-1}[\alpha+\log n+(2\alpha n/\log n)]$.

6. Consider now a real function $f(x_1, x_2, \cdots, x_p)$ defined for $-1\leqq x_k\leqq 1$, $k=1, 2, \cdots, p$. Let $n_1, n_2, \cdots, n_p$ be positive integers and set

$$\begin{aligned} &H_{n_1,\cdots,n_p}(f, x_1, x_2, \cdots, x_p) \\ &\equiv \sum_{h_1=1}^{n_1} \cdots \sum_{h_p=1}^{n_p} f(x_{h_1}^{(n_1)}, \cdots, x_{h_p}^{(n_p)}) A_{h_1}^{(n_1)}(x_1) \cdots A_{h_p}^{(n_p)}(x_p). \end{aligned} \tag{18}$$

The polynomial (18), which reduces to an Hermite-Fejér polynomial when $p=1$, has properties analogous to (4) and (5). Thus, for $k=1, 2, \cdots, p$, let j_k be a positive integer $\leqq n_k$. Then a repeated application of (4) yields easily

$$H_{n_1,n_2,\cdots,n_p}(f, x_{j_1}^{(n_1)}, \cdots, x_{j_p}^{(n_p)}) = f(x_{j_1}^{(n_1)}, \cdots, x_{j_p}^{(n_p)}).$$

Also, if $p\geqq 2$, and if $1\leqq k\leqq p$, $1\leqq j\leqq n_k$, then

$$\left(\frac{\partial}{\partial x_k} H_{n_1,n_2,\cdots,n_p}(f, x_1, \cdots, x_p)\right)_{x_k=x_j^{(n_k)}} \equiv 0. \tag{19}$$

Indeed, the left-hand side of (19) is identically equal to

$$\begin{aligned} &\sum_{h_q=1,2,\cdots,n_q;\, q=1,2,\cdots,p,\, q\neq k} \ \prod_{s=1;\, s\neq k}^{p} A_{h_s}^{(n_s)}(x_s) \\ &\qquad \cdot\left(\frac{d}{dx_k}\sum_{h_k=1}^{n_k} f(x_{h_1}^{(n_1)}, \cdots, x_{h_p}^{(n_p)}) A_{h_k}^{(n_k)}(x_k)\right)_{x_k=x_j^{(n_k)}} \end{aligned}$$

which is $\equiv 0$ by (1) and (5).

From (3) one gets

$$\sum_{h_1=1}^{n_1} \cdots \sum_{h_p=1}^{n_p} A_{h_1}^{(n_1)}(x_1) \cdots A_{h_p}^{(n_p)}(x_p) \equiv 1.$$

Consequently, by virtue of (18), if $p \geqq 2$,

$$\begin{aligned} &f(x_1, \cdots, x_p) - H_{n_1,\cdots,n_p}(f, x_1, \cdots, x_p) \\ &\equiv \sum_{h_1=1}^{n_1} \cdots \sum_{h_p=1}^{n_p} [f(x_1, \cdots, x_p) - f(x_{h_1}^{(n_1)}, \cdots, x_{h_p}^{(n_p)})] \\ &\qquad\qquad \cdot A_{h_1}^{(n_1)}(x_1) \cdots A_{h_p}^{(n_p)}(x_p) \\ &\equiv \sum_{h_1=1}^{n_1} \cdots \sum_{h_p=1}^{n_p} \sum_{r=1}^{p} [f(x_{h_1}^{(n_1)}, \cdots, x_{h_{r-1}}^{(n_{r-1})}, x_r, \cdots, x_p) \\ &\qquad\qquad - f(x_{h_1}^{(n_1)}, \cdots, x_{h_r}^{(n_r)}, x_{r+1}, \cdots, x_p)] \prod_{s=1}^{p} A_{h_s}^{(n_s)}(x_s) \\ &\equiv \sum_{r=1}^{p} \sum_{h_q=1,2,\cdots,n_q;\, q=1,2,\cdots,p,\, q\neq r} \Bigg\{ \sum_{h_r=1}^{n_r} [f(x_{h_1}^{(n_1)}, \cdots, x_{h_{r-1}}^{(n_{r-1})}, x_r, \cdots, x_p) \\ &\qquad - f(x_{h_1}^{(n_1)}, \cdots, x_{h_r}^{(n_r)}, x_{r+1}, \cdots, x_p)] A_{h_r}^{(n_r)}(x_r) \Bigg\} \prod_{s=1;\, s\neq r}^{p} A_{h_s}^{(n_s)}(x_s). \end{aligned}$$

Here and below $f(x_{h_1}^{(n_1)}, \cdots, x_{h_{r-1}}^{(n_{r-1})}, x_r, \cdots, x_p)$ means $f(x_1, \cdots, x_p)$ if $r=1$, and $f(x_{h_1}^{(n_1)}, \cdots, x_{h_r}^{(n_r)}, x_{r+1}, \cdots, x_p)$ means $f(x_{h_1}^{(n_1)}, \cdots, x_{h_p}^{(n_p)})$ if $r=p$.

7. THEOREM 3. *Let $f(x_1, x_2, \cdots, x_p)$ $(p \geqq 2)$ be a real function defined for $-1 \leqq x_k \leqq 1$, $k=1, 2, \cdots, p$. For $r=1, 2, \cdots, p$, let λ_r be a positive constant such that*

$$\begin{aligned} |f(x_1, \cdots, x_{r-1}, v, x_{r+1}, \cdots, x_p) - f(x_1, \cdots, x_{r-1}, u, x_{r+1}, \cdots, x_p)| \\ \leqq \lambda_r |v - u| \end{aligned}$$

whenever the x_q, u, and v are all in $[-1, 1]$. Let $n_1, n_2, \cdots, n_p$ be positive integers. Then

$$\begin{aligned} |f(x_1, x_2, \cdots, x_p) - H_{n_1,n_2,\cdots,n_p}(f, x_1, x_2, \cdots, x_p)| \\ < \sum_{r=1}^{p} 4\lambda_r \pi n_r^{-1}(\alpha + \log n_r) \end{aligned} \tag{20}$$

throughout the cube $-1 \leqq x_k \leqq 1$, $k=1, 2, \cdots, p$.

PROOF. Let $x_1, x_2, \cdots, x_p$ be points of $[-1, 1]$. We shall prove (20). For $r=1, 2, \cdots, p$, we have by (3), (1), and by Theorem 1,

$$\left| \sum_{h_r=1}^{n_r} \left[f(x_{h_1}^{(n_1)}, \cdots, x_{h_{r-1}}^{(n_{r-1})}, x_r, \cdots, x_p) - f(x_{h_1}^{(n_1)}, \cdots, x_{h_r}^{(n_r)}, x_{r+1}, \cdots, x_p) \right] A_{h_r}^{(n_r)}(x_r) \right|$$

$$< 4\lambda_r \pi n_r^{-1}(\alpha + \log n_r).$$

Since for $r = 1, 2, \cdots, p$,

$$\sum_{h_q=1,2,\cdots,n_q;\, q=1,2,\cdots,p,\, q\neq r} \ \prod_{s=1;\, s\neq r}^{p} A_{h_s}^{(n_s)}(x_s) = 1$$

and each summand is $\geqq 0$, therefore by the last paragraph of section 6. (20) holds.

Similarly, from Theorem 2 we obtain the following

THEOREM 4. *Let $f(x_1, x_2, \cdots, x_p)$ $(p \geqq 2)$ be a real function defined and bounded for $-1 \leqq x_k \leqq 1$, $k = 1, 2, \cdots, p$. For every $\delta \in [0, 2]$ and every r $(=1, 2, \cdots, p)$ let*

$$\omega_r(\delta) = \sup \left| f(x_1, \cdots, x_{r-1}, v, x_{r+1}, \cdots, x_p) - f(x_1, \cdots, x_{r-1}, u, x_{r+1}, \cdots, x_p) \right|,$$

where $x_1, \cdots, x_{r-1}, x_{r+1}, \cdots, x_p, u, v$ vary in $[-1, 1]$ with $0 \leqq v - u \leqq \delta$. Let $n_1, n_2, \cdots, n_p$ be positive integers $\geqq 2$. Then

$$\sup_{\substack{-1 \leqq x_k \leqq 1 \\ k=1,2,\cdots,p}} \left| f(x_1, x_2, \cdots, x_p) - H_{n_1, n_2, \cdots, n_p}(f, x_1, x_2, \cdots, x_p) \right|$$

$$\leqq \sum_{r=1}^{p} [2 + 4\pi + \epsilon_{n_r}]\omega_r(n_r^{-1} \log n_r),$$

where each ϵ_n $(n = 2, 3, \cdots)$ depends on n only, and $\epsilon_n \to 0$ as $n \to \infty$.

REFERENCES

1. L. Fejér, *Über Interpolation*, Göttinger Nachrichten (1916), 66–91.

2. E. Moldovan, *Observaţii asupra unor procedee de interpolare generalizate*, Acad. Repub. Pop. Romîne, Bul. Şti. Secţ. Şti. Mat. Fiz. 6 (1954), 477–482. (Romanian. French and Russian summaries)

3. O. Shisha, C. Sternin and M. Fekete, *On the accuracy of approximation to given functions by certain interpolatory polynomials of given degree*, Riveon Lematematika, 8 (1954), 59–64. (Hebrew. English summary)

AEROSPACE RESEARCH LABORATORIES,
WRIGHT-PATTERSON AIR FORCE BASE, OHIO