A REMARK ON THE EXISTENCE OF A G-STRUCTURE

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The purpose of this note is to show that a O-deformable tensor field \([4]\) defines a G-structure, i.e., a subbundle of the frame bundle.\(^{2}\)

We shall prove this statement in a slightly more general form. I am indebted to Professor K. Nomizu for calling my attention to this problem [Math. Reviews 27 (1964) #678], and to Professor T. Tamagawa for the valuable suggestion he made for the proof.

In [3, p. 294, Theorem], Crittenden shows that a cross section \(X\) of an associated bundle \((W, G, F, M, \pi)\) of a principal bundle \((P, G, M, \pi)\) is parallelizable if and only if \(f_X(P) = \beta_0 f\), where \(\beta_0 f\) is an orbit by the action \(\beta: G \times F \rightarrow F\), through some fixed \(f \in F\). Here \(f_X: P \rightarrow F\) is the differentiable map defined by \(f_X(p) = F(p)\left(X(\pi(p))\right)\) for \(p \in P\), where \(F(p): F \rightarrow F(\pi(p))\) is the map induced from \(P \times F\) to \(W\).

He further asserts that if \(X\) is parallelizable, \(B = f_X^{-1}(f)\) is a bundle with group \(K = \{g \in G | \beta_0 f = f\}\). In the proof of this assertion, the key idea is that \(f_X': P \rightarrow G/K\) determines a cross section in the associated bundle with fibre \(G/K\), where \(f_X'\) is defined by \(f_X' = \iota \circ f_X\), \(\iota\) being the map \(G/K \rightarrow \beta_0 f \subset F\) induced from \(g \rightarrow \beta_0 f\). However this \(f_X'\) is not necessarily differentiable.

This is precisely the point that Bernard worries about in the more special setting, where \(W\) is a tensor bundle (thus \(F\) is a vector space and \(G\) is the general linear group \(GL(n, \mathbb{R})\)), in [1, p. 211, Proposition III.2]. For \(f_X'\) to be differentiable, it suffices that \(f_X'\) be continuous, and for the latter it suffices that \((\iota, G/K)\) be a regular submanifold of \(F\).

Let us recall that "if \(G\) is a locally compact topological group which is a countable union of compact sets, \(S\) is a locally compact space, and if \(G\) acts on \(S\) as a transitive group of transformations then \(G/H_\pi\) is homeomorphic to \(S\), where \(H_\pi\) is the isotropy subgroup of \(G\) at a point \(p\) of \(S\)." The proof of this can be obtained from the proof in Pontrjagin [5, Theorem 12].

This shows, that, in order that \((\iota, G/K)\) be a regular submanifold of \(F\), it suffices that \(\beta_0 f\) be locally compact. In the rest we shall show the following lemma:

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2 This has been proved by Y. C. Wong [7, pp. 73–75]. We present a different proof.
Lemma. If \( F \) is the real vector space \( \mathbb{R}^n \), and \( G \) a real algebraic group contained in \( GL(n, \mathbb{R}) \), then \( \beta \) of is locally compact.

Corollary. Under the condition of the lemma, if \( X \) is parallelizable then \( B = f^1(\mathbb{R}) \) is a subbundle with group \( K \) of the principal bundle \( P \).

Proof of Lemma. By a real algebraic group \( G \) contained in \( GL(n, \mathbb{R}) \) we mean a group consisting of all invertible real \( n \times n \) matrices whose coefficients annihilate some set of polynomials with real coefficients in \( n^2 \) indeterminates. \( G \) is acting on \( \mathbb{R}^n \).

Let \( x_0 \in \mathbb{R}^n \) be fixed and consider the orbit \( G \cdot x_0 \). If \( G \) is irreducible (as an algebraic set) then \( G \cdot x_0 \) is also irreducible. If \( G \) is not irreducible, let the finite number of irreducible components of \( G \) be denoted by \( G_i \). If \( G_i \cdot x_0 \cap G_j \cdot x_0 \neq \emptyset \), then \( G_i \cdot x_0 = G_j \cdot x_0 \). Hence, in order to prove that \( G \cdot x_0 \) is locally compact (in the induced topology from the ordinary euclidean topology on \( \mathbb{R}^n \)), it suffices to assume \( G \) to be irreducible.

Now assuming \( G \) to be irreducible, let \( V \) be the smallest algebraic set in \( \mathbb{R}^n \) containing \( G \cdot x_0 \). \( V \) is irreducible. From [2, p. 191, Lemma 2, and p. 180, Proposition 13], we see that all the points of \( G \cdot x_0 \) are simple points of \( V \). By Whitney [6], we know that \( V = M_1 \cup V_1 \), where \( V_1 = V - M_1 \), \( V_1 \) is void or a proper algebraic set in \( V \), and \( M_1 \) is a manifold consisting of all simple points of \( V \).

Hence \( G \cdot x_0 \subset M_1 \). \( G \cdot x_0 \) is an open submanifold of \( M_1 \) (where we are considering the topology on \( M_1 \) induced from the ordinary euclidean topology of \( \mathbb{R}^n \)). Hence as \( G \cdot x_0 \) is an open set of \( M_1 \), which in turn is an open set of \( V \), which in turn is a closed set of \( \mathbb{R}^n \) with the euclidean topology, we conclude that \( G \cdot x_0 \) is locally compact. Q.E.D.

References


* The idea of this proof was taken from the proof of Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math. (2) 75 (1962), 485–535, p. 495, Proposition 2.3.
A NOTE ON A REDUCIBLE CONTINUUM

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In [4], Knaster shows that there exists an irreducible compact metric continuum $M$ which has a monotone continuous decomposition $G$ such that each element of $G$ is nondegenerate and $M/G$ is an arc. Also, he raised the question as to whether there existed an irreducible continuum $M$ which has a monotone continuous decomposition $G$ such that each element of $G$ is an arc and $M/G$ is an arc. E. E. Moise settled this question in the negative in [5]. In [3], M. E. Hamstrom showed that if $G$ is a monotone continuous decomposition of a compact metric continuum such that each element of $G$ is a non-degenerate continuous curve and $M/G$ is an arc, then it is not the case that $M$ is irreducible. E. Dyer generalized this result by showing in [2] that if $M$ is a compact metric continuum and $G$ is a monotone continuous decomposition of $M$ such that each element of $G$ is non-degenerate and decomposable, then it is not the case that $M$ is irreducible. A purpose of this note is to extend Dyer's result somewhat.

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**Theorem 1.** Let $M$ denote a compact metric continuum and $G$ a nondegenerate monotone continuous decomposition of $M$ each of whose elements is nondegenerate. If $H$ is a subcollection of $G$ each of whose elements is snakelike and indecomposable, and if $H^*$ is dense in $M$, then uncountably many elements of $G$ are indecomposable.

**Proof.** Let $I_1$ denote an element of $H$, and let $C_1$ denote the first chain in a sequence of defining chains for $I_1$, and let $L_1$ and $L_2$ denote the end links of $C_1$. Since $H^*$ is dense in $M$, and $G$ is a continuous collection, $C_1$ contains two elements $I(10)$ and $I(11)$ of $H$ such that $I(10)$ and $I(11)$ intersects every link of $C_1$. Let $\{C_n(10)\}$ and $\{C_n(11)\}$ denote chain sequences which define $I(10)$ and $I(11)$ respectively.

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