THE SUMMATION OF CERTAIN SERIES OF INFINITE REGRESSIVE ISOLS[1]

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1. **Introduction.** Denote the set of all non-negative integers by ϵ, the collection of all isols by Λ, and the collection of all regressive isols by Λ_R. If f is a function, we denote the range of f and domain of f by ρf and δf respectively. Dekker, in [1], defined and studied an infinite sum of non-negative integers. In this paper, we consider an infinite sum of infinite, regressive isols of the form $T-k$ for some $T\in\Lambda_R-\epsilon$.

2. **Summary.** We use the well-known pairing function $j(x, y)$ which maps ϵ^2 one to one onto ϵ and the functions $k(z)$, $l(z)$ such that $j(k(z), l(z))=z$. We also employ the mapping Φ_f, introduced in [6] as well as the partial ordering $\leqq*$ of Λ, defined in [2]. For $k\in\epsilon$ and t_n a regressive function, the set $(t_k, t_{k+1}, t_{k+2}, \cdots)$ is denoted by ρt_{n+k}.

DEFINITION. Let T and U be infinite, regressive isols and a_n a recursive function. Then

$$\sum_U (T - a_n) = \operatorname{Req} \bigcup_{k=0}^{\infty} j(u_k, \rho t_{n+a(k)}),$$

where u_k and t_n are any regressive functions ranging over sets in U and T respectively.

The principal results of this paper are as follows. Let a_n be a strictly increasing, recursive function. Then for T, $U\in\Lambda_R-\epsilon$,

$$\sum_U (T - a_n) = \sum_V (T - a_n), \quad \text{where} \quad V = \min(\Phi_a(T), U).$$

Moreover, with respect to the regressive isol V, the sum can be distributed over the difference $T-a_n$.

We note here several properties of the sum. It is readily shown that if t_n, t_n^* are any two regressive functions ranging over sets in T and u_k, u_k^* are any two regressive functions ranging over sets in U, then

$$\bigcup_{k=0}^{\infty} j(u_k, \rho t_{n+a(k)}) \simeq \bigcup_{k=0}^{\infty} j(\overset{*}{u_k}, \rho \overset{*}{t}_{n+a(k)}).$$

Presented to the Society, January 24, 1964; received by the editors June 12, 1964.

[1] The results in this paper are contained in the author's doctoral dissertation written at Rutgers, the State University, under the direction of Professor J. C. E. Dekker.

Hence the sum is uniquely defined. Moreover, the sum depends only on U and the infinite sequence of isols, $\{T-a_n\}$, and not on the choice of T itself, for it is easily proved that for U, $T\in\Lambda_R-\epsilon$ and k any integer such that $a_n+k\geqq 0$ for all n,

$$\sum_U ((T+k)-(a_n+k)) = \sum_U (T-a_n).$$

It is also apparent that for T, $U\in\Lambda_R-\epsilon$, the sum, $\sum_U (T-a_n)\in\Lambda$.

3. **Principal results.** If t_n is a regressive function having the regressing function $p(x)$, we make use of the partial recursive extension $p^*(x)$ of t^{-1}, defined by $p^*(x)=(\mu y)[p^{y+1}(x)=p^y(x)]$.

THEOREM 1. *Let a_n be a strictly increasing, recursive function. Then for T, $U\in\Lambda_R-\epsilon$,*

$$\sum_U (T-a_n) = \sum_V (T-a_n), \quad \textit{where} \quad V=\min(\Phi_a(T), U).$$

PROOF. We note that since a_n is strictly increasing and recursive, $\Phi_a(T)$ is a regressive isol and hence V is well defined and is also regressive. Let t_n, u_k be regressive functions ranging over sets in T and U respectively. By definition:

$$\sum_U (T-a_n) = \operatorname{Req} \bigcup_{k=0}^{\infty} j(u_k, \rho t_{n+a(k)}), \tag{1}$$

$$\sum_V (T-a_n) = \operatorname{Req} \bigcup_{k=0}^{\infty} j(j(t_{a(k)}, u_k), \rho t_{n+a(k)}). \tag{2}$$

Denote the sets appearing on the right in (1) and (2) by α and β respectively.

Let $p(x)$ and $q(x)$ be regressing functions of the regressive functions t_n and u_k respectively. Define

$$f(z) = j[j(p^{p^*l(z)-aq^*k(z)}l(z), k(z)), l(z)].$$

Let

$$g(z) = j(lk(z), l(z)).$$

Clearly, both f and g are partial recursive functions. For $z\in\alpha$, $p^*l(z)$ and $aq^*k(z)$ are defined, and $aq^*k(z)\leqq p^*l(z)$. Hence $\alpha\subset\delta f$. To verify that $f(\alpha)=\beta$, it is sufficient to note that for $z\in\alpha$, there exists m such that $k(z)=u_m$ and $l(z)=t_{a(m)+s}$ for some $s\in\epsilon$. Hence

$$p^{p^*l(z)-aq^*k(z)}l(z) = t_{a(m)}.$$

It readily follows that $f(\alpha)=\beta$. That f is 1-1 on α is a consequence of the fact that $j(x, y)$ is 1-1. Clearly, $\beta\subset\delta g$, $g(\beta)=\alpha$ and g is 1-1 on β. Furthermore, for $z\in\alpha$, $gf(z)=z$. An application of Proposition 1 of [1] completes the proof.

COROLLARY. *Let a_n be a strictly increasing, recursive function. Let $T, U, V\in\Lambda_R-\epsilon$. Then*

$$[\Phi_a(T) \leqq * U, \Phi_a(T) \leqq * V] \Rightarrow \sum_U (T - a_n) = \sum_V (T - a_n).$$

PROOF. Since $\Phi_a(T)\leqq * U$, $\min(\Phi_a(T), U)=\Phi_a(T)$. Since $\Phi_a(T)\leqq * V$, $\min(\Phi_a(T), V)=\Phi_a(T)$. The result follows by applying the theorem to both sums.

THEOREM 2. *Let a_n be a strictly increasing, recursive function. Then for $T, U\in\Lambda_R-\epsilon$,*

$$\sum_V (T - a_n) = TV - \sum_V a_n, \quad \textit{where} \quad V = \min(\Phi_a(T), U).$$

PROOF. It suffices to prove

$$\sum_V (T - a_n) + \sum_V a_n = VT. \tag{1}$$

Let t_n and u_k be regressive functions ranging over sets in T and U respectively. Let

$$\alpha = \bigcup_{k=0}^{\infty} j[j(t_{a(k)}, u_k), \rho t_{n+a(k)}],$$

$$\beta = \bigcup_{k=0}^{\infty} j[j(t_{a(k)}, u_k), \nu(a_k)],$$

$$\gamma = j[\rho j(t_{a(k)}, u_k), \rho t_n],$$

$$\delta = \bigcup_{k=0}^{\infty} j[j(t_{a(k)}, u_k), t\nu(a_k)].$$

Here, $\nu(a_k)$ denotes the set $(0, 1, \cdots, a_k-1)$. By definition, we have:

$$\sum_V (T - a_n) = \text{Req}\ \alpha,$$

$$\sum_V a_n = \text{Req}\ \beta,$$

$$VT = \text{Req}\ \gamma.$$

Since a_n is recursive, $\alpha | \delta$. We also have $\alpha + \delta = \gamma$. Hence to prove (1), it suffices to show that $\beta \simeq \delta$. Let $p(x)$ be a regressing function of the regressive function t_n and let $p^*(x)$ be related to $p(x)$ in the usual manner. Define

$$f(z) = j[k(z),\ p^{p^*kk(z)-l(z)}kk(z)],$$
$$g(z) = j(k(z),\ p^*l(z)).$$

Clearly, both f and g are partial recursive functions. Since for $z \in \beta$, $p^*kk(z) - l(z)$ is defined and $kk(z) \in \rho t_n$, we have $\beta \subset \delta f$. For $z \in \beta$, there exists m such that $kk(z) = t_{a(m)}$ and $l(z) = a_m - (s+1)$ for some $s \in \nu(a_m)$. Hence

$$p^{p^*kk(z)-l(z)}(kk(z)) = p^{a(m)-a(m)+s+1}(t_{a(m)}) = t_{a(m)-(s+1)}$$

and $f(\beta) = \delta$. Clearly f is 1-1 on β. The function $g(z)$ obviously has the properties:

$$\delta \subset \delta g, \qquad g(\delta) = \beta, \quad \text{and} \quad g \text{ is 1-1 on } \delta.$$

Since for $z \in \beta$, $gf(z) = z$, we have $\beta \simeq \delta$.

Combining the two preceding theorems, we obtain:

THEOREM 3. *Let a_n be a strictly increasing recursive function. Then for $T, U \in \Lambda_R - \epsilon$,*

$$\sum_U (T - a_n) = TV - \sum_V a_n, \quad \textit{where} \quad V = \min(\Phi_a(T), U).$$

The following are immediate corollaries of Theorem 3.

COROLLARY 1. *Let a_n be a strictly increasing, recursive function. Let $T, U \in \Lambda_R - \epsilon$. Then*

$$U \leqq * \Phi_a(T) \Rightarrow \sum_U (T - a_n) = TU - \sum_U a_n.$$

COROLLARY 2. *Let a_n be a strictly increasing, recursive function. Let $T, U \in \Lambda_R - \epsilon$. Then*

$$\Phi_a(T) \leqq * U \Rightarrow \sum_U (T - a_n) = T\Phi_a(T) - \sum_{\Phi_a(T)} a_n.$$

COROLLARY 3. *Let a_n be a strictly increasing, recursive function and let $T \in \Lambda_R - \epsilon$. Then*

$$\sum_T (T - a_n) = T\Phi_a(T) - \sum_{\Phi_a(T)} a_n.$$

4. REMARKS. We state here several other results whose hypotheses are more restrictive than those of Theorems 1 and 2. Their proofs will be omitted.

THEOREM 4. *Let* T, $U \in \Lambda_R - \epsilon$. *If* a_n *is a recursive function such that* $(\forall n)[a_n \leqq n+1]$, *then*

$$U \leqq {*}\, T \Rightarrow \sum_U (T - a_n) = TU - \sum_U a_n.$$

COROLLARY 1. *Let* $T \in \Lambda_R - \epsilon$. *If* a_n *is a recursive function satisfying the hypothesis of Theorem* 4, *then*

$$\sum_T (T - a_n) = T^2 - \sum_T a_n.$$

COROLLARY 2. *Let* $T \in \Lambda_R - \epsilon$. *Then*

$$\sum_T : T + (T - 1) + (T - 2) + \cdots = \sum_T : 1 + 2 + 3 + \cdots.$$

The second corollary can also be obtained by an application of Theorem 3.

For every increasing, unbounded, recursive function a_n, we define

$$\bar{a}_n = (\mu y)[a_y > n].$$

The function $\bar{a}_n$ is clearly partial recursive. Moreover, since a_n is unbounded, it follows that $\bar{a}_n$ is everywhere defined and hence recursive.

THEOREM 5. *Let* a_n *be an increasing, recursive function such that* $(\forall n)[a_n \geqq n]$. *Then for* T, $U \in \Lambda_R - \epsilon$,

$$T \leqq {*}\, U \Rightarrow \sum_U (T - a_n) = \sum_T \bar{a}_n.$$

COROLLARY. *Let* a_n *be an increasing, recursive function such that* $(\forall n)[a_n \geqq n]$. *Then for* $T \in \Lambda_R - \epsilon$,

$$\sum_T (T - a_n) = \sum_T \bar{a}_n.$$

Results similar to those in Theorems 1 and 2 can be obtained for sums whose terms consist of a product of factors. For T, U, $V \in \Lambda_R - \epsilon$ and a_n, b_n recursive functions, we define

$$\sum_V (T - a_n)(U - b_n) = \operatorname{Req} \bigcup_{k=0}^{\infty} j(v_k, j(\rho t_{n+a(k)}, \rho u_{n+b(k)})),$$

where t, u, v are regressive functions ranging over sets in T, U, V respectively.

THEOREM 6. *Let a_n and b_n be strictly increasing, recursive functions. Let T, U, $V \in \Lambda_R - \epsilon$ and $M = \min(\Phi_a(T), \Phi_b(U), V)$. Then*

$$\sum_V (T - a_n)(U - b_n) = \sum_M (T - a_n)(U - b_n).$$

Morever, with respect to the isol M, the sum can be distributed over the product to obtain $MTU - T\sum_M b_n - U\sum_M a_n + \sum_M a_n b_n$.

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