A CHARACTERIZATION OF ORBITS

JAMES W. ENGLAND

In this paper we give a necessary and sufficient condition in order that the orbit of a point be a homeomorphic image of the phase group in a transformation group.

The general reference for definitions is [2]. Throughout this paper (X, T, π) will denote a transformation group for which X is a first countable Hausdorff space. The phase group T will be assumed to be generative, that is, T is isomorphic to $C\times R^m\times I^n$ where C is a compact abelian group, R is the additive group of real numbers, I is the additive group of integers, and m and n are non-negative integers [4]. P will denote a replete semigroup in T which is distinct from T. Let $E\subset T$, then E is *P-extensive* provided $E\cap pP\neq\emptyset$ for each p in P. Let $x\in X$, then x is *P-recurrent* provided that for each neighborhood U of x there exists a P-extensive set E in T such that if $r\in E$ then $xr\in U$ [1]. For each $x\in X$ the isotropy subgroup T_x of x is the set of all $t\in T$ such that $xt=x$.

LEMMA 1. *If x is not P-recurrent for any replete semigroup $P\neq T$ then the isotropy subgroup T_x of x is a subgroup of the compact subgroup of T. That is, if $T=C\times R^m\times I^n$ then $T_x=A\times\{0\}\times\{0\}$ where $A\subset C$.*

PROOF. For $w\in T$ let $w=(w_1, w_2, w_3)$ where $w_1\in C$, $w_2\in R^m$ and $w_3\in I^n$. We assume that there exists an $a=(a_1, a_2, a_3)\in T_x$ such that not both a_2 and a_3 are zero. We consider first the case where $a_3\neq 0$. That is, if $a_3=(a_3(1), a_3(2), \cdots, a_3(n))$ then $a_3(j)\neq 0$ for some j. Let $P=C\times B$ where $B=\{(x_1, \cdots, x_{m+n})\in R^m\times I^n\colon x_{m+j}\geqq 2\}$. We first observe that P is a semigroup of T and $P\neq T$.

To see that P is replete let K be a compact subset of T. Then there exists a positive number r such that $C\times S(r)\supset K$. ($S(r)$ denotes the sphere of radius r about the origin.) It therefore follows that

$$\begin{aligned} &K(e', 0, \cdots, r+4, 0, \cdots, 0)\\ &\qquad\subset [C\times S(r)](e', 0, \cdots, r+4, 0, \cdots, 0)\subset P \end{aligned}$$

where $r+4$ appears in the $m+j+1$ position and e' denotes the identity in C. Thus P is a replete semigroup in T.

We now show that T_x is P-extensive. First, we observe that for $p\in P$ the set $pP=C\times B'$ where

$$B' = \{(x_1, \cdots, x_{m+n})\in R^m\times I^n\colon x_{m+j}\geqq N\}$$

Received by the editors April 27, 1965.

for some integer N. Also there exists an integer k such that $ka_3(j) > N$. Thus if we consider the element $ka = (ka_1, ka_2, ka_3)$ we have $ka_1 \in C$, $(ka_2, ka_3) \in B'$ hence $ka \in pP$. Also since T_x is a subgroup of T, $ka \in T_x$ which implies that $T_x \cap pP \neq \emptyset$ for all $p \in P$. Hence T_x is P-extensive.

For each $t \in T$, $xt = x$. Therefore, if U is any open set containing x then $xT_x \subset U$ which implies x is P-recurrent and is a contradiction. The case where $a_2 \neq 0$ can be handled in the same manner.

THEOREM 1. *A necessary and sufficient condition that the mapping $g: T \to 0(x)$ by $g(t) = xt$ be a homeomorphism is that the isotropy subgroup T_x of T at x restricted to the compact group C is trivial and that x is not P-recurrent for any replete semigroup $P \neq T$.*

PROOF. Assume that $g: T \to 0(x)$ is a homeomorphism. Since g is one-to-one it follows that T_x is trivial. In order to show that x is not P-recurrent for any replete semigroup of T we assume there exists a replete semigroup $P \neq T$ such that x is P-recurrent. Let $\{U_n\}$ be a sequence of open sets such that $x = \bigcap U_n$ and let A_n be the corresponding P-extensive subsets of T.

Let U be an open subset of C which contains e', the identity of C, and has the property that $U \neq C$. If S is a semigroup of R^1 which contains some interval containing zero then $S = R^1$. Thus, since $P \neq T$, for some j there exists a number p_j and an interval U_j containing zero such that if P_j is the projection of P onto the j-axis ($j > 1$) then $p_j \in P_j$ and $p_j P_j \cap U_j = \emptyset$. Let $K = U \times R^1 \times \cdots \times U_j \times \cdots \times I_n$ and $p = (x_1, \cdots, p_j, \cdots, x_{m+n+1}) \in P$. Then $pP \cap K = \emptyset$. Thus since $A_n \cap pP \neq \emptyset$ for all n we have $A_n \cap \{T - K\} \neq \emptyset$ for all integers n. Let $r_n \in A_n - K$ for each n. Since $\bigcap U_n = x$ it follows that $\lim_{n\to\infty} xr_n = xe = x$. But since g is a homeomorphism this implies that $\lim_{n\to\infty} r_n = e$ which is clearly impossible. Thus x is not P-recurrent for any $P \neq T$.

We now assume that x is not P-recurrent for any replete semigroup $P \neq T$. It follows from Lemma 1 that $T_x \subset C$. But since T_x restricted to C is trivial this means $T_x = \{(e', 0, 0)\}$. This implies that the mapping $g: T \to 0(x)$ by $g(t) = xt$ is one-to-one. Since (X, T, π) is a transformation group it follows that g is continuous. We have only to show that g^{-1} is continuous. In order to do this it is sufficient to show that if $\lim_{n\to\infty} xt_n = x$ then $\lim_{n\to\infty} t_n = e$.

If this is not the case then either there exists a subsequence $\{t_n'\}$ of $\{t_n\}$ and an $a = (a_1, a_2, a_3) \neq (e', 0, 0)$ such that $\lim_{n\to\infty} t_n' = a$ or the sequence $\{t_n\}$ intersects the set $T - \{C \times S(r)\}$ for all $r > 0$, where $S(r)$ denotes the sphere of radius r about the origin in $R^m \times I^n$. If

$\lim_{n\to\infty} t_n' = a$ then, since π is continuous, it follows that $xa = x$ which is a contradiction since g is one-to-one.

In the second case let $t_i = (t_i(1), \cdots, t_i(m+n+1))$ where $(t_i(2), \cdots, t_i(m+n+1)) \in R^m \times I^n$. For some $j > 1$ there exists a subsequence of $\{t_i(j)\}$ which converges either to positive or negative infinity. We consider the case where some subsequence of $\{t_i(j)\}$ converges to positive infinity and denote by t_i' the corresponding elements of $\{t_i\}$. The case where some subsequence of $\{t_i(j)\}$ converges to negative infinity can be handled in the same manner. If we let $P = C \times B$ where $B = \{(x_1, \cdots, x_{m+n}) \in R^m \times I^n : x_j \geqq 2\}$ then P is a replete semigroup of T and $\{t_i'\}$ is a P-extensive subset of T. It follows that x is P-recurrent, which is impossible.

We observe that if $m = n = 0$ then $T = C$. Since there are no replete semigroups of T other than T itself the notion of P-recurrence is not meaningful. In this case Theorem 1 reduces to the remarks found in [**3**, p. 65].

Bibliography

1. J. D. Baum, *P-recurrence in topological dynamics*, Proc. Amer. Math. Soc. **7** (1956), 1146–1154.

2. W. H. Gottschalk, and G. A. Hedlund, *Topological dynamics*, Amer. Math. Soc. Colloq. Publ. Vol. 36. Amer. Math. Soc., Providence, R. I., 1955.

3. D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, New York, 1955.

4. A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Actualités Sci. Ind., No. 869, Paris, 1940.

University of Virginia