NOTE ON ANALYTICALLY UNRAMIFIED SEMI-LOCAL RINGS

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All rings in this paper are assumed to be commutative rings with a unit element. If \( B \) is an ideal in a ring \( R \), the integral closure \( B_a \) of \( B \) is the set of elements \( x \) in \( R \) such that \( x \) satisfies an equation of the form \( x^n + b_1x^{n-1} + \cdots + b_n = 0 \), where \( b_i \in B_i \) \((i = 1, \cdots, n)\). An ideal \( B \) in \( R \) is semi-prime in case \( B \) is an intersection of prime ideals. If \( R \) is an integral domain, then \( R \) is normal in case \( R \) is integrally closed in its quotient field. If \( P \) is a semi-local (Noetherian) ring, then \( P \) is analytically unramified in case the completion of \( P \) (with respect to the powers of the Jacobson radical of \( R \)) contains no nonzero nilpotent elements.

Let \( R \) be a semi-local ring with Jacobson radical \( J \), and let \( R^* \) be the completion of \( R \). In [2], Zariski proved that if \( R \) is a normal local integral domain, and if there is a nonzero element \( x \) in \( J \) such that \( pR^* \) is semi-prime, for every prime divisor \( p \) of \( xR \), then \( R \) is analytically unramified. In [1, p. 132] Nagata proved that if \( R \) is a semi-local integral domain, and if there is a nonzero element \( x \) in \( J \) such that, for every prime divisor \( p \) of \( xR \), \( pR^* \) is semi-prime and \( R_p \) is a valuation ring, then \( R \) is analytically unramified. (The condition \( R_p \) is a valuation ring holds if \( R \) is normal.) The main purpose of this note is to extend Nagata's result to the case where \( R \) is a semi-local ring (Theorem 1). This extension will be given after first proving a

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number of lemmas. Among these preliminary results, Lemma 3 gives a necessary and sufficient condition for $R_p$ to be a discrete Archimedean valuation ring (where $R$ is a Noetherian ring and $p$ is a prime divisor of a nonzero-divisor $b \in R$), Corollary 2 of Lemma 5 gives a sufficient condition for a Noetherian ring to be a direct sum of normal Noetherian domains, and Lemma 6 gives a characterization of analytically unramified semi-local rings.

In Lemmas 1–4 below, $R$ is a Noetherian ring, $S$ is the integral closure of $R$ in its total quotient ring, $b$ is a nonunit in $R$ which is not a divisor of zero, $p$ is a prime divisor of $bR$, and $q$ is the isolated component of zero determined by $p$. If $B$ is an ideal in $R$, then $B^\prime R_p$ is the ideal generated by $(B+q)/q$ in $R_p$. Likewise, if $c \in R$, then $c'$ is the $q$-residue of $c$.

**Lemma 1.** $(bR)_a = bS \cap R$, and an element $c$ in $R$ is in $(bR)_a$ if and only if $c/b \in S$.

**Proof.** If $c \in (bR)_a$, then $c^n + b_1c^{n-1} + \cdots + b_n = 0$, where $b_i \in b^iR$. Dividing this equation by $b^n$ shows that $c/b \in S$, so $c \in bS \cap R$, hence $(bR)_a \subseteq bS \cap R$. If $c \in bS \cap R$, then $c/b \in S$, so $(c/b)^n + r_1(c/b)^{n-1} + \cdots + r_n = 0$, where $r_i \in R$. Multiplying this equation by $b^n$ shows that $c \in (bR)_a$, since $c \in R$. Therefore $bS \cap R \subseteq (bR)_a$, hence $(bR)_a = bS \cap R$, q.e.d.

**Lemma 2.** $R_p$ is a discrete Archimedean valuation ring if and only if $R_p$ is normal.

**Proof.** If $R_p$ is a valuation ring, then $R_p$ is normal. Conversely, if $R_p$ is normal, then $R_p$ is a normal local integral domain (hence, the kernel of the natural homomorphism from $R$ into $R_p$, which is $q$, is a prime ideal), and $p' R_p$ is a prime divisor of $b' R_p$. Since $b' R_p \neq (0)$, height $p' R_p = 1$ hence $R_p$ is a discrete Archimedean valuation ring [3, pp. 276–278], q.e.d.

An element $c \in R$ such that $bR : cR = p$ is used in the next lemma. Such an element can be found as follows. Let $p = p_1, p_2, \cdots, p_n$ be the prime divisors of $bR$, and let $d$ be an element in the $p_i$-primary component of $bR$ ($i = 2, \cdots, n$) which is not in $bR$. If $bR : dR \neq p$, let $e$ be an element in $(bR : dR) : pR$ which is not in $bR : dR$, and let $c = de$.

**Lemma 3.** Let $c$ be an element in $R$ such that $bR : cR = p$. $R_p$ is normal if and only if $c/b \in S$.

**Proof.** Let $R_p$ be normal. Since $bR : cR = p$, $b'R_p : c'R_p = p'R_p$. Therefore $c' \in b'R_p$, so $c'/b' \in R_p$. Hence, since $S_{R \setminus p}$ is contained in the integral closure of $R_p$ in its quotient field, $c/b \in S$. Conversely,
assume \( c/b \in S \). Since \( cp \subseteq bR \), \((c/b)p \subseteq R \). If \((c/b)p \subseteq p \), then \( bR[c/b] \subseteq pR[c/b] \subseteq R[c/b] \subseteq R \), so \( R[c/b] \) is contained in the finite \( R \)-module \((1/b)R \), hence \( c/b \in S \). This is a contradiction, so \( cp \not\subseteq bR \). Therefore, there are elements \( d \in p \), and \( x \in R \), \( \not\in p \), such that \( cd = bx \). Then \( b'R_p = b'x'R_p = c'd'x'R_p \subseteq c'p'R_p \subseteq b'R_p \), so \( c'p'R_p = b'R_p = c'd'R_p \). Now \( c' \) is not a divisor of zero in \( R_p \) (since \( b'x' \) is not), so \( p'R_p = (b'/c')R_p \), hence \( R_p \) is normal (Lemma 2, and [3, p. 277]), q.e.d.

**Lemma 4.** \( (bR)_a = bR \) if and only if \( R_p \) is normal, for every prime divisor \( p \) of \( bR \).

**Proof.** If \( R_p \) is normal, for every prime divisor \( p \) of \( bR \), then \( R_p = S_{bR \sim b} \), so \( p'R_p \cap S \) is a prime divisor of \( bS \). Let \( p_1, \ldots, p_n \) be the prime divisors of \( bR \), and let \( b_i \) be the image of \( b \) in \( R_{p_i} \). Then \( (bR)_a = bS \cap R \) (Lemma 1) \( \subseteq \bigcap_{i=1}^n (b_i R_{p_i} \cap S) \cap R = \bigcap_{i=1}^n (b_i R_{p_i} \cap R) = bR \subseteq (bR)_a \), hence \( (bR)_a = bR \). Conversely, let \( (bR)_a = bR \), let \( p \) be a prime divisor of \( bR \), and let \( c \) be an element in \( R \) such that \( bR : cR = p \). Then \( c/b \in R \). If \( R_p \) is not normal, then \( c/b \in S \) (Lemma 3), hence \( c \in bS \cap R = (bR)_a \) (Lemma 1). Since \( (bR)_a = bR \), this is a contradiction to \( c/b \in S \). Therefore \( R_p \) is normal, q.e.d.

**Lemma 5.** Let \( R \) be a Noetherian ring with Jacobson radical \( J \), let \( b \) be a nonzero element in \( J \), and let \( p_1, \ldots, p_n \) be the prime divisors of \( bR \). If \( R_{p_i} \) is a discrete Archimedean valuation ring \((i=1, \ldots, n)\), then the isolated component of zero contained in \( p_i \) is a prime ideal \( q_i \) and \( \bigcap_{i=1}^n q_i = (0) \). Moreover, \( b \) is not a zero-divisor in \( R \), and \( (bR)_a = bR \).

**Proof.** If \( R_{p_i} \) is a discrete Archimedean valuation ring, then \( R_{p_i} \) is an integral domain which is not a field, so the isolated component of zero contained in \( p_i \) is a prime ideal \( q_i \). Since \( q_i \) is the kernel of the natural homomorphism from \( R \) into \( R_{p_i} \), \( q_i \) is contained in every \( p_i \)-primary ideal. Hence, since \( bR = \bigcap_{i=1}^n (b_i R_{p_i} \cap R) \), where \( b_i \) is the \( q_i \)-residue of \( b \), and since each \( p_i \) is a minimal prime divisor of \( bR \), \( Z = \bigcap_{i=1}^n q_i \subseteq bR \). Since \( b \in q_i \) \((i=1, \ldots, n)\), \( Z : bR = Z \). This implies \( Z = bR \cap (Z : bR) = bZ : bR \). Therefore, since \( b \in J \), \( Z = b(Z : bR) = bZ \subseteq JZ \subseteq Z \). Hence, \( Z = \bigcap_{i=1}^n J^h Z \subseteq \bigcap_{i=1}^n J^h = (0) \). Thus \( b \) is not a zero-divisor, so \((bR)_a = bR \) (Lemma 4), q.e.d.

**Corollary 1.** With the same \( R \) and \( J \) of Lemma 5, suppose there is a nonzero nilpotent element in \( R \). If \( b \) is a nonzero divisor in \( J \), then \((bR)_a \neq bR \).

**Proof.** If \( b \) is a nonzero divisor in \( J \) such that \((bR)_a = bR \), then \( R_p \) is a discrete Archimedean valuation ring, for every prime divisor \( p \) of \( bR \) (Lemma 4). Hence by Lemma 5, the zero ideal in \( R \) is semiprime, q.e.d.
Corollary 4 below is the next result which is needed to prove Theorem 1, and it can be proved as a corollary to Lemma 5. Corollaries 1, 2, and 3, and Lemma 6 are not used in the proof of Theorem 1. They are included at this point because they are of some interest in themselves.

**Corollary 2.** Let $R$ be an integrally closed Noetherian ring, let $J$ be the Jacobson radical of $R$, and let $q_1, \ldots, q_n$ be the minimal prime divisors of zero. If there is a nonzero-divisor $b$ in $J$, then $R = \bigoplus_i R/q_i$, and $R/q_i$ is a normal Noetherian domain.

**Proof.** If $b$ is a nonzero-divisor in $J$, then $(bR)_a = bR$, since $R$ is semi-prime. Therefore by Corollary 1 the zero ideal in $R$ is semi-prime, and consequently the total quotient ring $Q$ of $R$ is the direct sum of $n$ fields. Since the idempotents in $Q$ are integrally dependent on $R$, they are in $R$. This, and the fact that $R$ is integrally closed, immediately imply the conclusions, q.e.d.

In Corollaries 3 and 4 and Lemmas 6 and 7, $R$ is a semi-local ring with maximal ideals $M_1, \ldots, M_d$, $J = \bigcap_i M_i$, and $R^*$ is the completion of $R$.

**Corollary 3.** Assume that no $M_i$ is a prime divisor of zero, and that $R^*$ is integrally closed. Then the completion of each $R_{M_i}$ is normal (hence $R_{M_i}$ is a normal local domain).

**Proof.** Since no $M_i$ is a prime divisor of zero, there is a nonzero-divisor $b$ in the Jacobson radical of $R^*$ [4, p. 267]. Hence by Corollary 2, $R^* = \bigoplus R^*/q_i$, where $q_i$ runs through the prime divisors of zero in $R^*$. Since the idempotents of the total quotient ring of $R^*$ are in $R^*$, no maximal ideal in $R^*$ contains more than one prime divisor of zero. Therefore, there are $d$ prime divisors of zero in $R^*$, since $R^*/q_i$ is a complete normal local domain. Let $M_iR^*$ be the maximal ideal in $R^*$ which contains $q_i$. Then it is immediately seen that $R^*_{M_i} = R^*/q_i \supseteq R/(q_i \cap R) = R_{M_i}$. Since $R_{M_i}$ is a dense subspace of $R^*/q_i$ [4, p. 283], the completion of $R_{M_i}$ is normal. It is well known [1, p. 59] that this implies that $R_{M_i}$ is a normal local domain, q.e.d.

**Lemma 6.** Let $b$ be a nonzero-divisor in $J$, let $R^*$ be the integral closure of $R^*$ in its total quotient ring, and let $T = R^* \cap R^*[1/b]$. If there is an integer $n$ such that $b^n \subseteq bR^*$, then $R$ is analytically unramified. Conversely, if $R$ is analytically unramified, then for every nonzero-divisor $c$ in $R$ there is an integer $k$ (depending on $c$) such that $c^k (R^* \cap R^*[1/c]) \subseteq cR^*$.

**Proof.** Since $b$ is not a divisor of zero in $R$, $b$ is not a divisor of zero in $R^*$ [4, p. 267], so $R^*[1/b]$ is contained in the total quotient
ring \( Q \) of \( R^* \). Let \( x \) be a nilpotent element in \( R^* \). Then \( x/b^i \in T \), for all \( i \geq 1 \). Therefore, if \( b^n T \subseteq b R^* \), then \( x \in b^i T \subseteq b^{i-n+1} R^* \subseteq J^{i-n+1} R^* \), for all \( i \geq n \). Since \( \bigcap J^i R^* = 0 \), \( x = 0 \). Hence \( R \) is analytically unramified. Conversely, let \( R \) be analytically unramified and let \( q_1, \ldots, q_n \) be the prime divisors of zero in \( R^* \). Then \( R^* = \bigoplus_1^n (R^*/q_i)' \), where \((R^*/q_i)'\) is the integral closure of \( R^*/q_i \). Since \((R^*/q_i)'\) is a finite \( R^*/q_i \)-module \([1, \text{p. 112}] \), \( R^* \) is a finite \( R^*-\)module. Thus \( R^* \cap R^*[1/c] \) is a finite \( R^*-\)module, for every non-zero-divisor \( c \) in \( R \). Hence, since every element in \( R^* \cap R^*[1/c] \) can be written in the form \( r/c^i \), where \( r \in (ci R^*)_a \), the last statement is clear, q.e.d.

**Corollary 4.** With the same notation as Lemma 6, assume \((b R^*)_a = b R^* \). Then \( R \) is analytically unramified.

**Proof.** If \( t \in T \), then \( t = r/b^i \), where \( r \in (b R^*)_a \). Since \( b R^* \) and \( b^i R^* \) have the same prime divisors, \((b R^*)_a = b R^* \) (Lemma 4). Therefore \( T = R^* \), hence \( b T = b R^* \), and so \( R \) is analytically unramified by Lemma 6, q.e.d.

**Lemma 7.** Let \( p \) be a height one prime ideal in \( R \). If \( R_p \) is normal, and if \( p R^* = \bigcap_1^h p_i^* \), where each \( p_i^* \) is a prime ideal in \( R^* \), then each \( R_p^* \) is normal, and \( p^{(n)} R^* = \bigcap_1^h p_i^{(n)} \) (where \( q^{(n)} \) is the \( n \)th symbolic power of a prime ideal \( q \)).

**Proof.** Since \( R_p \) is a normal local domain which is not a field, \( p \) is not a prime divisor of zero. Let \( b \) be an element in \( p \) such that \( b'R_p = p'R_p \) (\( B'R_p \) denotes the ideal in \( R_p \) generated by an ideal \( B \) in \( R \)). Then \( 0 : b R \subseteq q \), where \( q \) is the prime divisor of zero contained in \( p \). Therefore, \( (0 : b R) R^* = 0 R^* : b R^* \) \([4, \text{p. 267}] \) \( \subseteq q R^* \subseteq p R^* \subseteq p_i^* \) \((i = 1, \ldots, h) \). Fix \( i \), set \( p_i^* = p_i^* \), and let \( q^* \) be a prime divisor of \( 0 R^* \) which is contained in \( p_i^* \). Then \( q^* \cap R \) is a prime divisor of zero \([4, \text{p. 267}] \) and is contained in \( p = p^* \cap R \). Hence \( q^* \cap R = q \). Further, since \( q \) is the only \( q \)-primary ideal, every \( q \)-primary ideal contracts in \( R \) to \( q \). Hence \( R_p \) is a subring of \( R_p^* \), and, since \( 0 R^* : b R^* \subseteq q R^* \), \( b' \) is not a zero-divisor in \( R_p^* \). Since \( p R^* \) is semi-priime, \( b'R_p^* = p'R_p^* \) \( = p'' R_p^* \). Therefore \( R_p^* \) is normal (Lemma 2 and \([3, \text{pp. 276–278}] \)).

The proof that \( p^{(n)} R^* = \bigcap_1^h p_i^{(n)} \) is the same as that in \([2]\). Namely, since the result is true for \( n = 1 \), let \( n > 1 \) and assume \( p^{(n-1)} R^* = \bigcap_1^h p_i^{(n-1)} \). Let \( c \) be an element in \( b R : p \) which is not in \( p \) (since \( b'R_p = p'R_p, b : p \subseteq p \)), and let \( d^* \in \bigcap_1^h p_i^{(n)} \subseteq p R^* \). Since \( c \in b R^* : p R^* \), \( c d^* = b r^* \), for some \( r^* \in R^* \), hence by the choice of \( c \) and \( b \), \( b'r^* R_p^* = c'd'^* R_p^* = d'^* R_p^* \subseteq p_i^{(n)} R_p^* = b^i R_p^* \) \((i = 1, \ldots, h) \). Therefore, \( r^* \in \bigcap_1^h p_i^{(n-1)} \), so by induction \( r^* \in p^{(n-1)} R^* \). Thus \( c d^* = b r^* \in p^{(n)} R^* \), hence \( d^* \in p^{(n)} R^* : c R^* = (p^{(n)}) : c R) R^* \) \([4, \text{p. 267}] \) \( = p^{(n)} R^* \), since \( c \in p \).
Thus $\bigcap^n p_i^{(n)}(\mathfrak{p}^{(n)}R^*) \subseteq \mathfrak{p}^{(n)}R^*$, and since the opposite inclusion is clear, $\mathfrak{p}^{(n)}R^* = \bigcap^n p_i^{(n)}$, q.e.d.

**Theorem 1.** Let $R$ be a semi-local ring with Jacobson radical $J$, and let $R^*$ be the completion of $R$. Assume there is a non-zero-divisor $b$ in $R$ such that $(bR)_a = bR$ and $pR^*$ is semi-prime, for every prime divisor $p$ of $bR$. Then $(bR^*)_a = bR^*$. If $b \in J$, then $R$ is analytically unramified.

**Proof.** If $b$ is a unit in $R$, then $(bR^*)_a = bR^* = R^*$. Hence assume $b$ is a non-unit in $R$, and let $p_1, \ldots, p_h$ be the prime divisors of $bR$. Since each $R_{p_i}$ is a discrete Archimedean valuation ring (Lemmas 2 and 4), every $p_i$-primary ideal is a symbolic power of $p_i$. Therefore $bR = \bigcap^n p_i^{(e_i)}$, so $bR^* = \bigcap^n (p^{(e_i)}R^*)$ [4, p. 269]. Fix $i$, set $p^{(e)} = p_i^{(e_i)}$, and let $p_i^*, \ldots, p_h^*$ be the prime divisors of $pR^*$. Then $p^{(e)}R^* = \bigcap^n p_i^{(e_i)}$ and each $R_{p_i}^*$ is normal (Lemma 7). Thus the prime divisors of $bR^*$ are the prime divisors of the $p_iR^*$ ($i = 1, \ldots, h$), hence $(bR^*)_a = bR^*$ (Lemma 4). Therefore, if $b \in J$, then by Corollary 4, $R$ is analytically unramified, q.e.d.

**Corollary 5.** Let $R$, $R^*$ and $b$ be as in Theorem 1, and let $S^*$ be the integral closure of $R^*$ in its total quotient ring. If there is an element $v$ in $S^*$ such that $bv \in R^*$, then $v \in R^*$.

**Proof.** $bv \in bS^* \cap R^* = (bR^*)_a = bR^*$, q.e.d.

**References**


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