1. Introduction. In [1], Baumslag found all nilpotent standard restricted wreath products $A \wr B$. It is immediate that $A \wr B$ is solvable if and only if $A$ and $B$ are. In this paper we prove the following results.

**Theorem.** A finite wreath product $A \wr B$ is supersolvable if and only if:

(i) $A$ is nilpotent,
(ii) either $B$ is Abelian or $B'$ is a (nontrivial) $p$-group with $A$ a $p$-group for the same prime $p$, and
(iii) for each prime $q$ dividing $o(A)$, $q \equiv 1 \pmod{m}$, where $m$ is the exponent of $B/Q$, $Q$ being the Sylow $q$-subgroup of $B$ ($Q$ is unique by (ii), and may be trivial).

**Corollary.** If $A$ and $B$ are finite and satisfy conditions (i), (ii), and (iii) above, then any extension of $A$ by $B$ is supersolvable.

2. Notation and definitions. All groups considered are finite and are written multiplicatively. $E$ denotes the trivial group and 1 is used for the identity of all groups. Other notation is standard (as found, for example, in [8]).

Let $A$ and $B$ be nontrivial abstract groups, and let $F = A^B$ be the direct sum of copies of $A$ indexed by the set $B$. Explicitly, $F$ is the set of all functions from $B$ into $A$, made into a group by component-wise multiplication. For $f \in F$ and $b \in B$, define $f^b \in F$ by $f^b(y) = f(yb^{-1})$ for all $y \in B$. Then for each $b \in B$, the mapping $f \mapsto f^b$ is an automorphism of $F$, and the group of all such automorphisms is isomorphic to $B$. The (standard restricted) wreath product of $A$ and $B$, $A \wr B$, is defined to be the group generated by $F$ and $B$ with relations $b^{-1}f^b = f^b$ for all $f \in F, b \in B$. Here $F \rtimes A \wr B$, and $A \wr B$ is a splitting extension of $F$ by the group of automorphisms $B$. $W$ will be used throughout to designate $A \wr B$, $F$ (called the base group) will always be as defined above, and $K$ will denote the subgroup of $W$ defined by $K = \{f \in F : \Pi_{x \in B}(x) \in A'\}$.

We say that a group $G$ is supersolvable if and only if it has an invariant series whose factors are of prime order, that is, a series $G = G_n, G_{n-1}, \cdots, G_0 = E$ with each $G_i \triangleright G$ and each $G_{i+1}/G_i$ of prime
order (also called a supersolvable series). $G$ is nilpotent if and only if it has a series $G = G_m, G_{m-1}, \ldots, G_0 = E$ in which each $G_i \triangleleft G$ and each $G_{i+1}/G_i \subseteq Z(G/G_i)$. The properties of supersolvable and nilpotent groups needed can be found in [3] or [8]. Certain of these, as well as some results on wreath products, are listed for convenience in the following lemma.

Lemma 0.

(0.1) $W' = KB'$ [6, Corollary 4.5].

(0.2) If $C$ is a subgroup of $A$ and $D$ a subgroup of $B$, then $C \wr D$ can be embedded in $A \wr B$ [7].

(0.3) Every extension of $A$ by $B$ can be embedded in $A \wr B$ [5], [7].

(0.4) $A \wr B$ is nilpotent if and only if $A$ and $B$ are $p$-groups for the same prime $p$ [1].

(0.5) If $G$ is nilpotent then $G$ is supersolvable.

(0.6) If $G$ is supersolvable then $G'$ is nilpotent.

(0.7) Subgroups of nilpotent (supersolvable) groups are nilpotent (supersolvable).

(0.8) $G$ is nilpotent if and only if $H$ a proper subgroup of $G$ implies $H$ a proper subgroup of $N_o(H)$.

(0.9) $G$ is nilpotent if and only if it is the direct product of its Sylow subgroups.

3. The proofs.

Lemma 1. If $A \wr B$ is supersolvable, then $A$ is nilpotent.

Proof. If $A$ is not nilpotent, it contains a proper subgroup $H$ with $N_A(H) = H$ (0.8). Then $H_1 = \{f \in K : f(x) \in H \text{ for all } x \in B\}$ is a proper subgroup of $K \subseteq W'$ (0.1). Suppose $f \in N_K(H_1) \setminus H_1$. Then there exists $x \in B$ such that $f(x) = a \in A \setminus H$. If we choose $y \in B$, $y \neq x$, and define $g \in H_1$ by $g(x) = h$, $g(y) = h^{-1}$, and $g(z) = 1$ for $z \neq x, y$, where $h$ is any element of $H$, we see that $f^{-1}gf \in H_1$, so that $(f^{-1}gf)(x) = a^{-1}ha \in H$. But since $h$ is arbitrary in $H$ this implies $a \in N_A(H) \setminus H$, which is impossible. Therefore $N_K(H_1) = H_1$, contradicting the nilpotency of $K$ (0.8) and $W'$ (0.7) and therefore the supersolvability of $W$ (0.6).

Lemma 2. If $A \wr B$ is supersolvable, then $B$ is Abelian or $B'$ is a (nontrivial) $p$-group and $A$ is a $p$-group for the same prime $p$.

Proof. If the lemma is false, then there are primes $p \mid o(B')$, $q \mid o(A)$, $p \neq q$. Take $a \in A$ with $o(a) = q$ and $b \in B'$ with $o(b) = p$. Choose arbitrary distinct elements $x, y \in B$, and define $f \in F$ by $f(x) = a$, $f(y) = a^{-1}$, $f(z) = 1$ for $z \neq x, y$. Then $f \in K \subseteq W'$, $b \in W'$, $o(f) = q$, $o(b) = p$, but $fb \neq bf$. (If $a = a^{-1}$ for all $a \in A$, then $o(B) > 2$ and
x and y can be chosen so that we have \( fb \neq bf \). Therefore \( W' \) is not nilpotent (0.9) and so \( W \) is not supersolvable (0.6).

**Lemma 3.** \( A \wreath B \) is supersolvable if and only if \( A \) is nilpotent and \( J_p \wreath B \) is supersolvable for each \( p \) dividing \( o(A) \), where \( J_p \) denotes the cyclic group of order \( p \).

**Proof.** If \( A \) is not nilpotent, \( W \) is not supersolvable by Lemma 1, while if \( p \mid o(A) \) and \( J_p \wreath B \) is not supersolvable, then \( A \wreath B \) is not supersolvable by (0.2) and (0.7).

For sufficiency, induct on \( o(A) \). There is a subgroup \( H \) of \( Z(A) \) of prime order. Since \( H \wreath B \) is supersolvable, there is an invariant series of \( W \) up through \( H^B \) with factors of prime order. Then \( W/H \cong (A/H) \wreath B \) is supersolvable by induction, and therefore \( W \) is also supersolvable.

**Lemma 4.** If \( p \nmid n \), \( p \) prime, then \( J_p \wreath \mathcal{G} \) is supersolvable if and only if \( p \equiv 1 \pmod{n} \).

**Proof.** Since \( \mathcal{G} \) is supersolvable and \( W/F \cong J_n \), and \( F \) is Abelian, it is necessary and sufficient to show that \( F \) has a series \( F = F_n, F_{n-1}, \ldots, F_0 = E \) with each \( o(F_{i+1}/F_i) = p \) and each \( F_i \) invariant under the action of \( B = J_n \). Here we may view \( F \) as a vector space of dimension \( n \) over \( GF(p) \), a field with \( p \) elements, acted on by the cyclic linear transformation \( b \) induced by \( b \), where \( \langle b \rangle = B = J_n \). The minimum polynomial of \( b \) is \( \lambda^n - 1 \), and the subspaces invariant under \( b \) may be put in one-one correspondence with the factors of \( \lambda^n - 1 \) which have leading coefficient 1, with one such subspace containing another if and only if the factor corresponding to the first divides the factor corresponding to the second [4, p. 129]. Therefore \( J_p \wreath J_n \) is supersolvable if and only if \( \lambda^n - 1 \) splits into linear factors over \( GF(p) \). Since \( p \nmid n \), \( \lambda^n - 1 \) has no repeated roots in \( GF(p) \), and so \( \lambda^n - 1 \) splits into linear factors if and only if \( n \mid (p-1) \), for the roots of \( \lambda^n - 1 \) form a subgroup of the multiplicative group of \( GF(p) \).

**Lemma 5.** If \( B \) is Abelian and \( p \nmid o(B) \), then \( J_p \wreath B \) is supersolvable if and only if \( p \equiv 1 \pmod{m} \), where \( m = \text{Exp } B \).

**Proof.** If \( p \equiv 1 \pmod{m} \), we may apply (0.2), (0.7), and Lemma 4 to conclude that \( J_p \wreath B \) is not supersolvable since \( J_p \wreath J_m \) is not. As in the proof of Lemma 4, it is now sufficient to show that if \( p \equiv 1 \pmod{m} \) then \( F \) has a series \( F = F_n, F_{n-1}, \ldots, F_0 = E \) with each \( o(F_{i+1}/F_i) = p \) and each \( F_i \) invariant under the action of \( B \), where in the present case \( n = o(B) \). We interpret \( B \) as a group acting on \( F \), a vector space of dimension \( n \) over \( GF(p) \), and apply Maschke's
Theorem [2, pp. 40–41] to conclude that $F$ is a direct sum of irreducible $B$-subspaces. If all of these irreducible $B$-subspaces are of dimension one, it is clear how to construct the sequence $F = F_n, F_{n-1}, \ldots, F_0 = E$. However, since $p \mid \text{Exp } B$, it follows [2, Theorem 9.10, p. 37] that $B$ has $o(B)$ distinct one-dimensional $GF(p)$-representations, and therefore all irreducible $B$-subspaces are one-dimensional [2, p. 213]. (Since $p \equiv 1 \pmod{m}$, $\lambda^m - 1$ splits into linear factors over $GF(p)$, and so in the notation of [2], $\sqrt{1} \in GF(p)$.)

**Lemma 6.** If $P$ is a Sylow $p$-subgroup of $B$ containing $B'$, then $J_p \wr B$ is supersolvable if and only if $p \equiv 1 \pmod{m}$, where $m = \text{Exp } (B/P)$.

**Proof.** Suppose $J_p \wr B$ is supersolvable. Since $P \triangleleft B$, $P$ has a complement $Q$ in $B$ by Schur's splitting theorem [8, 9.3.6]. Then $J_p \wr Q$ is supersolvable by (0.2) and (0.7), and hence $p \equiv 1 \pmod{m}$ by Lemma 5. Now assume $p \equiv 1 \pmod{m}$. We can use the fact that $P \triangleleft B$ to conclude that $FP \triangleleft W$, and thereby think of $W$ as an extension of $FP$ by $B/P$. Then Lemmas 3 and 5 and (0.3) and (0.7) apply to yield that $J_p \wr B$ is supersolvable.

**Proof of Theorem.** If $A \wr B$ is supersolvable, the necessity of (i) follows from Lemma 1, that of (ii) from Lemma 2, and that of (iii) from Lemma 3 and Lemma 6. Sufficiency is shown by Lemmas 3, 5, 6, and (0.4).

**Proof of Corollary.** Apply the Theorem, (0.3), and (0.7).

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**References**


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