BLOCK IDEMPOTENTS OF TWISTED GROUP ALGEBRAS[1]

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In [4] Conlon has successfully generalized much of the theory of modular representations to the projective case. However his generalization [4, p. 166] of one of Brauer's main theorems on blocks [3, 10B], [5] is not entirely satisfactory. In Theorem 1 we present another generalization which is closer than Conlon's to the original Brauer theorem, and in Theorem 2 we indicate an application involving the number of blocks with a given defect group.

Let G be a finite group and Ω a field of prime characteristic p. A *twisted group algebra* $\Gamma(G)$ of G over Ω is an associative Ω-algebra with a basis consisting of elements (g) in one-to-one correspondence with the elements g of G, with multiplication determined by equations

$$(g)(h) = \epsilon_{g,h}(gh), \qquad g, h \in G,$$

where $0 \neq \epsilon_{g,h} \in \Omega$. By associativity, $\epsilon = \{\epsilon_{g,h}\}$ must be a factor set of G in Ω. It is well known that the projective representations of G in Ω with factor set ϵ can be identified with the representations of $\Gamma(G)$ [6].

For any $g \in G$, define

$$C^{\epsilon}(g) = \{x \in G\colon (x)^{-1}(g)(x) = (g)\}.$$

It is evident that $C^{\epsilon}(g)$ is a subgroup of the centralizer $C(g)$ of g in G. Let us call g *ϵ-regular* provided that $C^{\epsilon}(g) = C(g)$. A short calculation shows that

$$(1) \qquad C^{\epsilon}(h^{-1}gh) = h^{-1}C^{\epsilon}(g)h, \qquad g, h \in G;$$

hence the set of all ϵ-regular elements is a union of conjugate classes of G, which we call the *ϵ-regular classes* of G.

We assume[2] that $\Gamma(G)$ satisfies the following conditions:

$$(2) \qquad (h)^{-1}(g)(h) = (h^{-1}gh), \qquad g, h \in G,\ g\ \epsilon\text{-regular};$$

$$(3) \qquad (g^{-1}) = (g)^{-1}, \qquad g \in G.$$

(Condition (2) is never an essential restriction; and neither is (3) if Ω is algebraically closed [4, §1].)

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[2] In fact we do not need to assume (3), since it is not required in the proof of Conlon's theorem.

For each ϵ-regular class K, let $(K) = \sum_{g\in K} (g)$; these *ϵ-regular class sums* (K) form a basis of the center $\Lambda(G)$ of $\Gamma(G)$. As usual, we call any p-Sylow subgroup of $C(g)$ for any $g \in K$ a *defect group* of K. For any *block idempotent*, i.e. primitive idempotent, e of $\Lambda(G)$, write $e = \sum_K f_K(K)$, $f_K \in \Omega$. Then the largest of the defect groups of the K for which $f_K \neq 0$ can be called a *defect group* of e; this is uniquely determined up to conjugacy in G [**4**, §3].

Let D be an arbitrary p-subgroup of G. Let $C(D)$ be the centralizer of D in G, and denote the normalizer $N(D)$ of D in G by H. For each ϵ-regular class K, set

$$s((K)) = \sum_{g\in K\cap C(D)} (g).$$

By [**4**, §3], extending s by linearity gives an Ω-algebra homomorphism $s: \Lambda(G) \to \Lambda(H)$, where $\Lambda(H)$ is the center of the twisted group algebra $\Gamma(H)$ of H whose factor set is the restriction $\epsilon|H$ of ϵ to H. Adapting our terminology to H in the obvious way, we can now state:

THEOREM 1. *The homomorphism s determines a one-to-one correspondence $e \leftrightarrow s(e)$ between the block idempotents of $\Lambda(G)$ which have D as one of their defect groups and the block idempotents of $\Lambda(H)$ which have D as their unique defect group.*

We shall show that Theorem 1 follows from Conlon's theorem. The lgtter states that $e \leftrightarrow s(e)$ is a one-to-one correspondence between the block idempotents of $\Lambda(G)$ which have D as one of their defect groups and the primitive idempotents of $U(D)$, where $U(D)$ is a subalgebra of $\Lambda(H)$ which has as a basis those $(\epsilon|H)$-regular class sums (L) of H such that L has defect group D and consists of ϵ-regular elements. (Since only ϵ-regular elements are involved, these class sums are defined in $\Lambda(H)$, even though the analogue of (2) for $\epsilon|H$ need not hold.) Furthermore each primitive idempotent of $U(D)$ is a sum of block idempotents of $\Lambda(H)$ which have defect group D.

As Conlon points out, the complication in his theorem is due to the fact that an $(\epsilon|H)$-regular element need not be ϵ-regular. However, we can prove:

LEMMA. *Every $(\epsilon|H)$-regular element whose conjugate class in H has defect group D is ϵ-regular.*

This lemma implies that the $(\epsilon|H)$-regular class sums (L) of H such that L has defect group D form a basis of $U(D)$, and hence that $U(D)$ contains all block idempotents of $\Lambda(H)$ with defect group D. Therefore these idempotents of $\Lambda(H)$ are precisely all the primitive idempotents of $U(H)$. This proves that Theorem 1 follows from Conlon's theorem.

It remains to prove the lemma. Let $h\in H$ satisfy the hypothesis of the lemma. Since h is $(\epsilon|H)$-regular, $C(h)\cap H\subseteq C^{\epsilon}(h)$. Since D is the unique defect group of the class of h in H, D is a p-Sylow subgroup of $C(h)\cap H$. Then the second paragraph of the proof of [5, Lemma 3.4] shows that D is a p-Sylow subgroup of $C(h)$, and hence also of $C^{\epsilon}(h)$. For any $x\in C(h)$, $x^{-1}Dx$ is a p-Sylow subgroup of $x^{-1}C^{\epsilon}(h)x$, which equals $C^{\epsilon}(h)$ by (1). Then $x^{-1}Dx=y^{-1}Dy$ for some $y\in C^{\epsilon}(h)$, and $xy^{-1}\in N(D)\cap C(x)=H\cap C(x)\subseteq C^{\epsilon}(h)$. Hence $x\in C^{\epsilon}(h)$, so that h is ϵ-regular as required.

Theorem 1 can be applied in conjunction with the methods of Bovdi [1] to generalize [1, Theorems 1 and 2] as follows (cf. [2, Corollary 1]).

THEOREM 2. *The number of block idempotents of $\Lambda(G)$ with D as a defect group is less than or equal to the number of p-regular ϵ-regular classes K of G with D as a defect group such that (K) is not a nilpotent element of $\Lambda(G)$.*

Equality holds here if G has a normal subgroup T of p-power index such that T has a normal p-Sylow subgroup, while Ω is algebraically closed.

In a later paper we shall give a proof of a more general form of Theorem 2.

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