BLOCK IDEMPOTENTS OF TWISTED GROUP ALGEBRAS

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In [4] Conlon has successfully generalized much of the theory of modular representations to the projective case. However his generalization [4, p. 166] of one of Brauer's main theorems on blocks [3, 10B], [5] is not entirely satisfactory. In Theorem 1 we present another generalization which is closer than Conlon's to the original Brauer theorem, and in Theorem 2 we indicate an application involving the number of blocks with a given defect group.

Let $G$ be a finite group and $\Omega$ a field of prime characteristic $p$. A twisted group algebra $\Gamma(G)$ of $G$ over $\Omega$ is an associative $\Omega$-algebra with a basis consisting of elements $(g)$ in one-to-one correspondence with the elements $g$ of $G$, with multiplication determined by equations

$$ (g)(h) = \epsilon_{\theta,h}(gh), \quad g, h \in G, $$

where $0 \neq \epsilon_{\theta,h} \in \Omega$. By associativity, $\epsilon = \{\epsilon_{\theta,h}\}$ must be a factor set of $G$ in $\Omega$. It is well known that the projective representations of $G$ in $\Omega$ with factor set $\epsilon$ can be identified with the representations of $\Gamma(G)$ [6].

For any $g \in G$, define

$$ C^*(g) = \{x \in G : (x)^{-1}(g)(x) = (g)\}. $$

It is evident that $C^*(g)$ is a subgroup of the centralizer $C(g)$ of $g$ in $G$. Let us call $g \epsilon$-regular provided that $C^*(g) = C(g)$. A short calculation shows that

$$ C^*(h^{-1}gh) = h^{-1}C^*(g)h, \quad g, h \in G; $$

hence the set of all $\epsilon$-regular elements is a union of conjugate classes of $G$, which we call the $\epsilon$-regular classes of $G$.

We assume\(^2\) that $\Gamma(G)$ satisfies the following conditions:

\begin{align*}
(2) \quad (h)^{-1}(g)(h) &= (h^{-1}gh), \quad g, h \in G, \ g \ \epsilon\text{-regular}; \\
(3) \quad (g^{-1}) &= (g)^{-1}, \quad g \in G.
\end{align*}

(Condition (2) is never an essential restriction; and neither is (3) if $\Omega$ is algebraically closed [4, §1].)

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\(^2\) In fact we do not need to assume (3), since it is not required in the proof of Conlon's theorem.
For each \( \epsilon \)-regular class \( K \), let \( (K) = \sum_{g \in K} (g) \); these \( \epsilon \)-regular class sums \( (K) \) form a basis of the center \( \Lambda(G) \) of \( T(G) \). As usual, we call any \( p \)-Sylow subgroup of \( C(g) \) for any \( g \in K \) a defect group of \( K \). For any block idempotent, i.e. primitive idempotent, \( e \) of \( \Lambda(G) \), write 
\[
e = \sum_{K} f_{K}(K), f_{K} \in \Omega.\]
Then the largest of the defect groups of the \( K \) for which \( f_{K} \neq 0 \) can be called a defect group of \( e \); this is uniquely determined up to conjugacy in \( G \) [4, §3].

Let \( D \) be an arbitrary \( p \)-subgroup of \( G \). Let \( C(D) \) be the centralizer of \( D \) in \( G \), and denote the normalizer \( N(D) \) of \( D \) in \( G \) by \( \mathcal{N} \). For each \( \epsilon \)-regular class \( K \), set
\[
s((K)) = \sum_{g \in K \cap C(D)} (g).\]
By [4, §3], extending \( s \) by linearity gives an \( \Omega \)-algebra homomorphism \( s: \Lambda(G) \rightarrow \Lambda(H) \), where \( \Lambda(H) \) is the center of the twisted group algebra \( T(H) \) of \( H \) whose factor set is the restriction \( \epsilon | H \) of \( \epsilon \) to \( H \). Adapting our terminology to \( H \) in the obvious way, we can now state:

**Theorem 1.** The homomorphism \( s \) determines a one-to-one correspondence \( e \leftrightarrow s(e) \) between the block idempotents of \( \Lambda(G) \) which have \( D \) as one of their defect groups and the block idempotents of \( \Lambda(H) \) which have \( D \) as their unique defect group.

We shall show that Theorem 1 follows from Conlon’s theorem. The letter states that \( e \leftrightarrow s(e) \) is a one-to-one correspondence between the block idempotents of \( \Lambda(G) \) which have \( D \) as one of their defect groups and the primitive idempotents of \( U(D) \), where \( U(D) \) is a subalgebra of \( \Lambda(H) \) which has as a basis those \( (\epsilon | H) \)-regular class sums \( (L) \) of \( H \) such that \( L \) has defect group \( D \) and consists of \( \epsilon \)-regular elements. (Since only \( \epsilon \)-regular elements are involved, these class sums are defined in \( \Lambda(H) \), even though the analogue of (2) for \( \epsilon | H \) need not hold.) Furthermore each primitive idempotent of \( U(D) \) is a sum of block idempotents of \( \Lambda(H) \) which have defect group \( D \).

As Conlon points out, the complication in his theorem is due to the fact that an \( (\epsilon | H) \)-regular element need not be \( \epsilon \)-regular. However, we can prove:

**Lemma.** Every \( (\epsilon | H) \)-regular element whose conjugate class in \( H \) has defect group \( D \) is \( \epsilon \)-regular.

This lemma implies that the \( (\epsilon | H) \)-regular class sums \( (L) \) of \( H \) such that \( L \) has defect group \( D \) form a basis of \( U(D) \), and hence that \( U(D) \) contains all block idempotents of \( \Lambda(H) \) with defect group \( D \). Therefore these idempotents of \( \Lambda(H) \) are precisely all the primitive
idempotents of $U(H)$. This proves that Theorem 1 follows from Conlon's theorem.

It remains to prove the lemma. Let $h \in H$ satisfy the hypothesis of the lemma. Since $h$ is $(e\mid H)$-regular, $C(h) \cap H \subseteq C'(h)$. Since $D$ is the unique defect group of the class of $h$ in $H$, $D$ is a $p$-Sylow subgroup of $C(h) \cap H$. Then the second paragraph of the proof of [5, Lemma 3.4] shows that $D$ is a $p$-Sylow subgroup of $C(h)$, and hence also of $C'(h)$. For any $x \in C(h)$, $x^{-1}Dx$ is a $p$-Sylow subgroup of $x^{-1}C'(h)x$, which equals $C'(h)$ by (1). Then $x^{-1}Dx = y^{-1}Dy$ for some $y \in C'(h)$, and $xy^{-1} \in N(D) \cap C(x) = H \cap C(x) \subseteq C'(h)$. Hence $x \in C'(h)$, so that $h$ is $e$-regular as required.

Theorem 1 can be applied in conjunction with the methods of Bovdi [1] to generalize [1, Theorems 1 and 2] as follows (cf. [2, Corollary 1]).

Theorem 2. The number of block idempotents of $\Lambda(G)$ with $D$ as a defect group is less than or equal to the number of $p$-regular $e$-regular classes $K$ of $G$ with $D$ as a defect group such that $(K)$ is not a nilpotent element of $\Lambda(G)$.

Equality holds here if $G$ has a normal subgroup $T$ of $p$-power index such that $T$ has a normal $p$-Sylow subgroup, while $\Omega$ is algebraically closed.

In a later paper we shall give a proof of a more general form of Theorem 2.

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References

1. A. A. Bovdi, The number of blocks of characters of a finite group with a given defect, Ukrain. Mat. Ž. 13 (1961), 136–141. (Russian)

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