ON THE SHEFFER A-TYPE OF POLYNOMIALS GENERATED BY $A(t)\psi[xB(t)]$

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Huff and Rainville [1] have proved: *If* $\{p_n(x)\}$ *is generated by* $A(t)\psi[xt]$ *then a necessary and sufficient condition that* $\{p_n(x)\}$ *be a Sheffer* A*-type* $m>0$ *is*

(1) $\psi(xt) = {}_0F_m[-;b_1,\cdots;b_m;\alpha xt]$, α *a nonzero constant.*

(For a discussion of the properties of Sheffer A-type sets and sets of Rainville σ-type zero, we refer the reader to Rainville [2, Chapter 13]. All results not specifically referenced can be found in this work.) It is the purpose of this note to generalize the Huff-Rainville theorem by establishing a complete characterization of the Sheffer A-type $m>0$ sets which are Boas and Buck sets.

Toward this end, suppose

$$\sum_{n=0}^{\infty} p_n(x)t^n = A(t)\psi[xB(t)] \tag{2}$$

with

$$\sum_{n=0}^{\infty} \psi_n t^n = \psi(t) \qquad \psi_n \neq 0,$$

$$\sum_{n=0}^{\infty} \alpha_n t^n = A(t) \qquad \alpha_0 \neq 0,$$

and

$$\sum_{n=0}^{\infty} \beta_n t^{n+1} = B(t) \qquad \beta_0 \neq 0.$$

Because $\psi_n \neq 0$ $(n \geqq 0)$ we are assured that $\{p_n(x)\}$ is a simple set of polynomials; specifically $p_n(x) = a_n x^n + O(x^{n-1})$ with $a_n \neq 0$ $(n \geqq 0)$. Associated with $\{p_n(x)\}$ is the unique differential operator $J(x, D)$, defined by the condition $J(x, D)p_n(x) = p_{n-1}(x)$, $n = 1, 2, \cdots$, where

$$J(x, D) = \sum_{n=0}^{\infty} T_n(x) D^{n+1},$$

$D \equiv d/dx$ and $T_n(x) = t_n x^n + O(x^{n-1})$ a polynomial of degree $\leqq n$. Since

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$a_0 \neq 0$, we conclude that $t_0 \neq 0$. Let $B^{-1}(t)$ be the formal power series inverse of $B(t)$. We state our main result as

THEOREM A. *If* $\{p_n(x)\}$ *is defined by* (2), *then a necessary and sufficient condition for* $\{p_n(x)\}$ *to be Sheffer A-type* $m > 0$ *is that there exist a positive integer* r *which divides* m *and numbers* $b_1, \cdots, b_r$ *not zero nor negative integers such that*

$$\psi[xB(t)] = {}_0F_r[-; b_1, \cdots, b_r; \alpha x B(t)] \tag{3}$$

for some nonzero constant α, *with* $B^{-1}(t)$ *a polynomial of degree* $s = m/r$, *exactly.*

PROOF. Suppose that $\{p_n(x)\}$ is Sheffer A-type m. The expression $J(x, D)p_n(x) = p_{n-1}(x)$ implies the recurrence relation

$$a_n(nt_0 + n(n-1)t_1 + \cdots + n(n-1)\cdots(n-m)t_m) = a_{n-1} \tag{4}$$

for $n = 1, 2, 3, \cdots$ obtained by equating coefficients of x^{n-1}. Since the coefficient of $n \cdot a_n$ in (4) is a polynomial in $(n-1)$ of degree r with $1 \leq r \leq m$, factorization yields the recurrence relation

$$cna_n \prod_{k=1}^{r} (n + b_k - 1) = a_{n-1} \tag{5}$$

where $c \neq 0$. Notice that $b_k = 0, -1, -2, \cdots$, for any k $(1 \leq k \leq r)$ would imply $a_i = 0$ for some i. We have previously remarked that $a_n \neq 0$ $(n \geq 0)$ hence b_k is neither zero nor a negative integer for any k. Equation (5) may be solved for a_n in terms of a_0 and yields

$$a_n = \frac{c^{-n} a_0}{n! \prod_{k=1}^{r} (b_k)_n}$$

where $(b_k)_n = b_k(b_k + 1) \cdots (b_k + n - 1)$. In the proof of Theorem 49, [2, p. 141], it is shown that $a_n = a_0 \beta_0^n \psi_n$. Thus

$$\psi_n = \frac{(c\beta_0)^{-n}}{n! \prod_{k=1}^{r} (b_k)_n}$$

and hence $\psi(t) = {}_0F_r[-; b_1, \cdots, b_r; t/c\beta_0]$. Then from (2)

$$\sum_{n=0}^{\infty} p_n(x)t^n = A(t)\,{}_0F_r[-; b_1, \cdots, b_r; \alpha x B(t)] \tag{6}$$

where $\alpha = (c\beta_0)^{-1} \neq 0$. To complete the proof of the necessity there remains to show that $m/r=s$ is an integer, and that $B^{-1}(t)$ is a polynomial of degree s. Now (6) is seen to imply that $\{p_n(x)\}$ is σ-type zero with $\sigma=D\prod_{k=1}^{r}(xD+b_k-1)$. (See [2, p. 228].) Hence there exists $J^*(\sigma)$ such that

$$(7)\quad J^*(\sigma)p_n(x) = \sum_{k=0}^{\infty}\gamma_k\left\{D\prod_{i=1}^{r}(xD+b_i-1)\right\}^{k+1}p_n(x) = p_{n-1}(x)$$

for $n=1, 2, \cdots$. Now (7) may be rearranged by collecting powers of D into $J(x, D)$, since $J(x, D)$ is unique. That is, $J^*(\sigma)p_n(x)=p_{n-1}(x)$ and $J(x, D)p_n(x)=p_{n-1}(x)$ imply $J^*(\sigma)=J(x, D)$. A simple check of (7) proves that $J(x, D)$ will contain polynomials $T_k(x)$ (as coefficient of D^{k+1}) with degree exactly m and no higher only if $kr=m$ for one k (say $k=s$), so that $\gamma_{s-1}\neq 0$ and $\gamma_s=\gamma_{s+1}=\cdots=0$. In view of this and (7), we have

$$J^*(t) = \sum_{k=0}^{s-1}\gamma_k t^{k+1}.$$

But $J^*(t)=B^{-1}(t)$, [2, Theorem 79], so that $B^{-1}(t)$ is a polynomial of degree $s=m/r$. This completes the proof of the necessity. Now suppose that there exists a positive integer r which divides m and numbers $b_1, \cdots, b_r$ so that (3) holds for some nonzero constant α with $B^{-1}(t)$ a polynomial of degree $m/r=s$, exactly. We need to show that $\{p_n(x)\}$ is Sheffer A-type m. But these hypotheses imply $\{p_n(x)\}$ is σ-type zero with $\sigma\equiv D\prod_{k=1}^{r}(xD+b_k-1)$. Since $J^*(t)=B^{-1}(t)$, we have

$$(8)\quad \begin{aligned}\sum_{k=0}^{s-1}\gamma_k&\left\{D\prod_{i=1}^{r}(xD+b_i-1)\right\}^{k+1}p_n(x)\\ &= \sum_{k=0}^{rs+s-1}T_k(x)D^{k+1}p_n(x) = p_{n-1}(x)\qquad (s\geqq 1)\end{aligned}$$

for $n=1, 2, \cdots$. A detailed check of the left-most expression in (8) will verify that $T_{rs+s-1}(x)$ is of degree rs exactly and that $T_k(x)$ is always of degree $\leqq rs$. The middle term in (8) is $J(x, D)p_n(x)$ and hence $\{p_n(x)\}$ is Sheffer A-type $rs=m$. This completes the proof.

We have remarked in the course of the proof of Theorem A that (6) implies $\{p_n(x)\}$ is σ-type zero for $\sigma\equiv D\prod_{k=1}^{r}(xD+b_k-1)$. Conversely, if $\{p_n(x)\}$ is σ-type zero for this σ, then (6) holds. We may thus re-word Theorem A as follows:

THEOREM B. *A necessary and sufficient condition that* $\{p_n(x)\}$, *de-*

fined by (2), *is Sheffer A-type* $m>0$ *is that there exists a positive integer* r *which divides* m *and numbers* $b_1, \cdots, b_r$, (*none zero nor a negative integer*) *such that* $\{p_n(x)\}$ *is* σ*-type zero for*

$$\sigma \equiv D \prod_{k=1}^{r} (xD + b_k - 1)$$

and $B^{-1}(t)$ *is a polynomial of degree* $s = m/r$ *exactly.*

REMARK. The choice $s=1$ reduces Theorem A to the Huff-Rainville result since $B^{-1}(t)$ is of degree one in this case.

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REFERENCES

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2. E. D. Rainville, *Special functions*, Macmillan, New York, 1960.

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