ON THE SHEFFER $A$-TYPE OF POLYNOMIALS
GENERATED BY $A(t)\psi[tB(t)]$

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Huff and Rainville [1] have proved: If $\{p_n(x)\}$ is generated by $A(t)\psi[t]$, then a necessary and sufficient condition that $\{p_n(x)\}$ be a Sheffer $A$-type $m > 0$ is

\[(1) \quad \psi(zt) = \psi_m(\alpha t) = oF_m[- ; b_1, \ldots ; b_m; \alpha x t], \quad \alpha \neq 0 \text{ a nonzero constant.}\]

(For a discussion of the properties of Sheffer $A$-type sets and sets of Rainville $\sigma$-type zero, we refer the reader to Rainville [2, Chapter 13]. All results not specifically referenced can be found in this work.) It is the purpose of this note to generalize the Huff-Rainville theorem by establishing a complete characterization of the Sheffer $A$-type $m > 0$ sets which are Boas and Buck sets.

Toward this end, suppose

\[(2) \quad \sum_{n=0}^{\infty} p_n(x)t^n = A(t)\psi[tB(t)]\]

with

\[\sum_{n=0}^{\infty} \psi_n t^n = \psi(t) \quad \psi_n \neq 0,\]

\[\sum_{n=0}^{\infty} \alpha_n t^n = A(t) \quad \alpha_0 \neq 0,\]

and

\[\sum_{n=0}^{\infty} \beta_n t^{n+1} = B(t) \quad \beta_0 \neq 0.\]

Because $\psi_n \neq 0$ ($n \geq 0$) we are assured that $\{p_n(x)\}$ is a simple set of polynomials; specifically $p_n(x) = a_n x^n + O(x^{n-1})$ with $a_n \neq 0$ ($n \geq 0$). Associated with $\{p_n(x)\}$ is the unique differential operator $J(x, D)$, defined by the condition $J(x, D)p_n(x) = p_{n-1}(x)$, $n = 1, 2, \ldots$, where

\[J(x, D) = \sum_{n=0}^{\infty} T_n(x) D^{n+1},\]

$D \equiv d/dx$ and $T_n(x) = t_n x^n + O(x^{n-1})$ a polynomial of degree $\leq n$. Since

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Let \( a_0 \neq 0 \), we conclude that \( t_0 \neq 0 \). Let \( B^{-1}(t) \) be the formal power series inverse of \( B(t) \). We state our main result as

**Theorem A.** If \( \{ p_n(x) \} \) is defined by (2), then a necessary and sufficient condition for \( \{ p_n(x) \} \) to be Sheffer A-type \( m > 0 \) is that there exist a positive integer \( r \) which divides \( m \) and numbers \( b_1, \ldots, b_r \) not zero nor negative integers such that

\[
\psi[xB(t)] = {}_0F_r[-; b_1, \ldots, b_r; \alpha xB(t)]
\]

for some nonzero constant \( \alpha \), with \( B^{-1}(t) \) a polynomial of degree \( s = m/r \), exactly.

**Proof.** Suppose that \( \{ p_n(x) \} \) is Sheffer A-type \( m \). The expression \( J(x, D)p_n(x) = p_{n-1}(x) \) implies the recurrence relation

\[
a_n(nt_0 + n(n-1)t_1 + \cdots + n(n-1)\cdots(n-m)t_m) = a_{n-1}
\]

for \( n = 1, 2, 3, \ldots \) obtained by equating coefficients of \( x^{n-1} \). Since the coefficient of \( n \cdot a_n \) in (4) is a polynomial in \( n-1 \) of degree \( r \) with \( 1 \leq r \leq m \), factorization yields the recurrence relation

\[
c_n a_n \prod_{k=1}^{r} (n + b_k - 1) = a_{n-1}
\]

where \( c \neq 0 \). Notice that \( b_k = 0, -1, -2, \ldots \), for any \( k \) \( (1 \leq k \leq r) \) would imply \( a_i = 0 \) for some \( i \). We have previously remarked that \( a_n \neq 0 \) \( (n \geq 0) \) hence \( b_k \) is neither zero nor a negative integer for any \( k \). Equation (5) may be solved for \( a_n \) in terms of \( a_0 \) and yields

\[
a_n = \frac{c^{-n}a_0}{n! \prod_{k=1}^{r} (b_k)_n}
\]

where \( (b_k)_n = b_k(b_k+1)\cdots(b_k+n-1) \). In the proof of Theorem 49, [2, p. 141], it is shown that \( a_n = a_0 b_n^c \psi_n \). Thus

\[
\psi_n = \frac{(c \beta_0)^{-n}}{n! \prod_{k=1}^{r} (b_k)_n}
\]

and hence \( \psi(t) = {}_0F_r[-; b_1, \ldots, b_r; t/c \beta_0] \). Then from (2)

\[
\sum_{n=0}^{\infty} p_n(x)t^n = A(t) {}_0F_r[-; b_1, \ldots, b_r; \alpha xB(t)]
\]

where \( \alpha = (c \beta_0)^{-1} \neq 0 \). To complete the proof of the necessity there re-
mains to show that $m/r = s$ is an integer, and that $B^{-1}(t)$ is a polynomial of degree $s$. Now (6) is seen to imply that $\{p_n(x)\}$ is $\sigma$-type zero with $\sigma = D \prod_{k=1}^{r} (xD + b_k - 1)$. (See [2, p. 228].) Hence there exists $J^*(\sigma)$ such that

$$(7) \quad J^*(\sigma)p_n(x) = \sum_{k=0}^{\infty} \gamma_k \left\{ D \prod_{i=1}^{r} (xD + b_i - 1) \right\}^{k+1} p_n(x) = p_{n-1}(x)$$

for $n = 1, 2, \cdots$. Now (7) may be rearranged by collecting powers of $D$ into $J(x, D)$, since $J(x, D)$ is unique. That is, $J^*(\sigma)p_n(x) = p_{n-1}(x)$ and $J(x, D)p_n(x) = p_{n-1}(x)$ imply $J^*(\sigma) = J(x, D)$. A simple check of (7) proves that $J(x, D)$ will contain polynomials $T_k(x)$ (as coefficient of $D^{k+1}$) with degree exactly $m$ and no higher only if $kr = m$ for one $k$ (say $k = s$), so that $\gamma_{s-1} \neq 0$ and $\gamma_s = \gamma_{s+1} = \cdots = 0$. In view of this and (7), we have

$$J^*(t) = \sum_{k=0}^{r-1} \gamma_k t^{k+1}.$$

But $J^*(t) = B^{-1}(t)$, [2, Theorem 79], so that $B^{-1}(t)$ is a polynomial of degree $s = m/r$. This completes the proof of the necessity. Now suppose that there exists a positive integer $r$ which divides $m$ and numbers $b_1, \cdots, b_r$ so that (3) holds for some nonzero constant $\alpha$ with $B^{-1}(t)$ a polynomial of degree $m/r = s$, exactly. We need to show that $\{p_n(x)\}$ is Sheffer $A$-type $m$. But these hypotheses imply $\{p_n(x)\}$ is $\sigma$-type zero with $\sigma = D \prod_{i=1}^{r} (xD + b_k - 1)$. Since $J^*(t) = B^{-1}(t)$, we have

$$(8) \quad \sum_{k=0}^{r-1} \gamma_k \left\{ D \prod_{i=1}^{r} (xD + b_i - 1) \right\}^{k+1} p_n(x) = \sum_{k=0}^{rs+s-1} T_k(x) D^{k+1} p_n(x) = p_{n-1}(x) \quad (s \geq 1)$$

for $n = 1, 2, \cdots$. A detailed check of the left-most expression in (8) will verify that $T_{rs+s-1}(x)$ is of degree $rs$ exactly and that $T_k(x)$ is always of degree $\leq rs$. The middle term in (8) is $J(x, D)p_n(x)$ and hence $\{p_n(x)\}$ is Sheffer $A$-type $rs = m$. This completes the proof.

We have remarked in the course of the proof of Theorem A that (6) implies $\{p_n(x)\}$ is $\sigma$-type zero for $\sigma = D \prod_{i=1}^{r} (xD + b_k - 1)$. Conversely, if $\{p_n(x)\}$ is $\sigma$-type zero for this $\sigma$, then (6) holds. We may thus re-word Theorem A as follows:

**Theorem B.** A necessary and sufficient condition that $\{p_n(x)\}$, de-
fined by (2), is Sheffer A-type $m > 0$ is that there exists a positive integer $r$ which divides $m$ and numbers $b_1, \ldots, b_r$, (none zero nor a negative integer) such that $\{p_n(x)\}$ is $\sigma$-type zero for

$$\sigma = D \prod_{k=1}^{r} (xD + b_k - 1)$$

and $B^{-1}(t)$ is a polynomial of degree $s = m/r$ exactly.

REMARK. The choice $s = 1$ reduces Theorem A to the Huff-Rainville result since $B^{-1}(t)$ is of degree one in this case.

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REFERENCES


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