ON THE SOLUTIONS OF THE DIFFERENTIAL EQUATION $y''+p^2y=0$

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If $p(x)(x \geqq 0)$ satisfies the conditions

(H_1) *p is positive, and its derivative p' is non-negative and continuous and* $\lim_{x\to\infty} p(x) = \infty$,

it is well-known that each solution of the equation

$$y'' + p(x)^2 y = 0 \qquad (x \geqq 0) \tag{1}$$

has infinitely many zeros, only finitely many on each interval $[0, x_0]$; also, $|y|$ is bounded and the values of $|y(x)|$ at successive maxima form a descending sequence. However, it does not follow from (H_1) that

$$\lim_{x\to\infty} y(x) = 0 \tag{2}$$

(cf. Galbraith, McShane and Parrish, [**3**]). The purpose of this note is to find hypotheses which, added to (H_1), insure that (2) holds. Roughly, the added hypotheses are to insure that p does not do essentially all of its increasing on a set of intervals over which $\int p\,dx$ is small and essentially nonincreasing over a set over which that integral is large.

THEOREM. *Let p: $[0, \infty)\to R$ satisfy* (H_1) *and*

(H_2) *there exists a positive ϵ such that for every sequence*

$$a_1 < b_1 < c_1 < a_2 < b_2 < c_2 < \cdots \tag{3}$$

such that

$$\int_{b_j}^{c_j} p(x)\,dx > \pi - \epsilon, \quad \int_{c_j}^{a_{j+1}} p(x)\,dx < \epsilon, \quad \int_{a_j}^{b_j} (b_j - x)p(x)^2\,dx < \epsilon^2 \tag{4}$$

it is true that

$$\sum_{j=1}^{\infty} [\log p(c_j) - \log p(b_j)] = \infty. \tag{5}$$

Then every solution of (1) *satisfies* (2).

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Suppose that (H_1) holds and that y satisfies (1). For $x \geqq 0$ define

$$r(x) = [y(x)^2 + y'(x)^2/p(x)^2]^{1/2}; \tag{6}$$

then except at the zeros of y',

$$p(x) = |y'(x)| [r(x)^2 - y(x)^2]^{-1/2}. \tag{7}$$

From (6) and (1),

$$r(x)r'(x) = -y'(x)^2 p'(x)/p(x)^3, \tag{8}$$

so that r is nonincreasing.

If $r(x) \to 0$ as $x \to \infty$, (2) holds. We shall assume the contrary and prove that then (H_2) is not satisfied. Without loss of generality we may assume

$$\lim_{x \to \infty} r(x) = 1. \tag{9}$$

Let ϵ be any positive number less than $1/2$, and δ a positive number less than $\epsilon/2$ such that

$$\arccos[(1 - \delta^2)/(1 + \delta^2)]^{1/2} < \epsilon/2. \tag{10}$$

By (9), there is an $\bar{x}$ such that if $x > \bar{x}$ then

$$1 \leqq r(x) < (1 + \delta^2)^{1/2}. \tag{11}$$

Let $a_1, a_2, a_3, \cdots$ be the successive extrema of $y(x)$ on $(\bar{x}, \infty)$. In $[a_j, a_{j+1}]$ there is a single zero z_j of y; by (6) and (11),

$$|y'(z_j)| / p(z_j) \geqq 1.$$

Let $[b_j, c_j]$ be the largest subinterval of $[a_j, a_{j+1}]$ that contains z_j and has

$$|y'(x)| / p(x) \geqq \delta \qquad (b_j \leqq x \leqq c_j). \tag{12}$$

To be specific, we assume $y' > 0$ on (a_j, a_{j+1}); discussion of the other case requires only replacement of y by $-y$. By continuity, equality holds in (12) at b_j and at c_j, so by (6) and (11)

$$y(c_j) \geqq (1 - \delta^2)^{1/2}, \qquad y(b_j) \leqq -(1 - \delta^2)^{1/2}. \tag{13}$$

Since r is nonincreasing and (11) holds, by (7) we have

$$p(x) \geqq y'(x)[1 + \delta^2 - y(x)^2]^{-1/2} \qquad (b_j \leqq x \leqq c_j),$$
$$p(x) \leqq y'(x)[r(a_{j+1})^2 - y(x)^2]^{-1/2} \qquad (c_j \leqq x \leqq a_{j+1}).$$

We integrate these over the intervals indicated and apply (13) and (10), obtaining

$$\begin{aligned}\int_{b_j}^{c_j} p(x)\,dx &\geqq \arccos[-(1-\delta^2)/(1+\delta^2)]^{1/2} \\ &\quad - \arccos[(1-\delta^2)/(1+\delta^2)]^{1/2} \\ &> \pi - \epsilon, \\ \int_{c_j}^{a_{j+1}} p(x)\,dx &\leqq \arccos[y(c_j)/r(a_{j+1})] \\ &< \arccos(1-\delta^2)^{1/2} \\ &< \epsilon/2.\end{aligned}$$

So the first and second estimates in (4) hold.

By (6) and (13),

$$y(b_j) - y(a_j) \leqq -(1-\delta^2)^{1/2} + (1+\delta^2)^{1/2} < 2\delta^2 < \epsilon^2/2,$$

while if $a_j \leqq v \leqq b_j$,

$$y(v) \leqq y(b_j) \leqq -[1-(1/4)^2]^{1/2} < -1/2.$$

This last implies

$$y''(v) > \tfrac{1}{2}p(v)^2 \qquad (a_j \leqq v \leqq b_j).$$

Hence, by an integration by parts,

$$\begin{aligned}\epsilon^2 > 2[y(b_j) - y(a_j)] &= 2\int_{a_j}^{b_j}\left[\int_{a_j}^{\xi} y''(v)\,dv\right]d\xi \\ &\geqq \int_{a_j}^{b_j}\left[\int_{a_j}^{\xi} p(v)^2\,dv\right]d\xi \\ &= \int_{a_j}^{b_j}(b_j-\xi)p(\xi)^2\,d\xi,\end{aligned}$$

and the third estimate in (4) holds.

However, by (11), (8) and (12),

$$\begin{aligned}(1+\delta^2) - 1 &> -\int_{\bar{x}}^{\infty} 2r(x)r'(x)\,dx \\ &\geqq \sum_{j=1}^{\infty}\int_{b_j}^{c_j}[-2r(x)r'(x)]\,dx = 2\sum_{j=1}^{\infty}\int_{b_j}^{c_j} y'(x)^2p'(x)p(x)^{-3}\,dx \\ &\geqq 2\delta^2\sum_{j=1}^{\infty}\int_{b_j}^{c_j}[p'(x)/p(x)]\,dx = 2\delta^2\sum_{j=1}^{\infty}[\log p(c_j) - \log p(b_j)],\end{aligned}$$

so the sum in the last expression cannot exceed 1/2, and (H_2) is not satisfied.

From the theorem we now deduce some corollaries in which the hypotheses, though stronger, are less intricate.

COROLLARY 1. *Let* (H_1) *be satisfied, and also*

(H_3) *there exists a positive number* δ *such that for every sequence*

$$0 < b_1 < c_1 < b_2 < c_2 < \cdots \tag{14}$$

such that

$$\int_{b_j}^{c_j} p(x)\,dx > \pi - \delta \quad \textit{and} \quad b_{j+1} - c_j < \delta/p(c_j) \tag{15}$$

equation (5) *holds.*

Then every solution of (1) *tends to zero as* $x \to \infty$.

Let the sequence (3) satisfy (4) with $\epsilon = \min(\delta, 1)/3$. Since p is nondecreasing, from (4) we obtain

$$p(c_j)(a_{j+1} - c_j) < \epsilon, \qquad \tfrac{1}{2}(b_{j+1} - a_{j+1})^2 p(a_{j+1})^2 < \epsilon^2,$$

whence (15) holds. So (5) holds, and (H_2) is satisfied. By the theorem, the conclusion follows.

More specially, if (H_1) holds, and there is a $\delta > 0$ such that (5) holds whenever the sequence (14) satisfies

$$\sum_{j=1}^{n} (b_{j+1} - c_j)/b_{n+1} < \delta \tag{16}$$

for all large n, every solution (1) tends to zero as $x \to \infty$. This theorem was stated by G. Armellini [1], but his proof seems incomplete.

COROLLARY 2. *Let* (H_1) *be satisfied, and also*

(H_4) *there exist positive numbers* K, δ, $\bar{x}$ *such that for every triple of numbers* x_1, x_2, x_3 *such that* $x_2 > \bar{x}$ *and*

$$x_2 - \delta/p(x_2) \leqq x_1 \leqq x_2 \leqq x_3 \leqq x_2 + \delta/p(x_2) \tag{17}$$

it is true that

$$p'(x_3)/p(x_3) \leqq K p'(x_1)/p(x_1).$$

Then every solution of (1) *tends to zero as* $x \to \infty$.

We may suppose $\delta < 1$. Let the sequence (14) satisfy (15). If we define $\epsilon_j = \delta/p(c_j)$ we have

$$b_j < c_j - \epsilon_j, \quad b_{j+1} < c_j + \epsilon_j,$$

and

$$\begin{aligned}\log p(c_j) - \log p(b_j) &\geqq \int_{c_j-\epsilon_j}^{c_j} [p'(x)/p(x)]\,dx \\ &\geqq K^{-1}\int_{c_j}^{c_j+\epsilon_j} [p'(x)/p(x)]\,dx \\ &\geqq K^{-1}[\log p(b_{j+1}) - \log p(c_j)].\end{aligned}$$

Hence

$$\log p(b_{j+1}) - \log p(b_j) \leqq (1+K)[\log p(c_j) - \log p(b_j)],$$

and the series (5) diverges. By Corollary 1, the conclusion follows.

Corollary 2 includes the theorems by Biernacki [2] and Milloux [5], which are somewhat too lengthy to state here.

COROLLARY 3. *Let* (H_1) *be satisfied, and also*

(H_5) *there exist positive numbers* K, $\bar{x}$, δ *such that if* $x_3 > x_1 \geqq \bar{x}$,

$$p(x_3)^{-2}p'(x_3) \leqq Kp(x_1)^{-2}p'(x_1). \tag{18}$$

Then every solution of (1) *tends to zero as* $x \to \infty$.

Define $H = Kp(\bar{x})^{-2}p'(\bar{x})$; by (18), if $x \geqq \bar{x}$

$$p(x)^{-2}p'(x) \leqq H.$$

Let x_1, x_2, x_3 satisfy (17) with $\delta = 1/4H$. Then

$$p(x_1)^{-1} - p(x_3)^{-1} = \int_{x_1}^{x_3} p(x)^{-2}p'(x)\,dx \leqq H(x_3 - x_1),$$

whence

$$\left[\frac{p(x_2)}{p(x_1)} - 1\right] + \left[1 - \frac{p(x_2)}{p(x_3)}\right] \leqq H(x_3 - x_1)p(x_2) \leqq 1/2.$$

Both terms in the left member are nonnegative, so each is at most 1/2. That is,

$$p(x_2) \leqq (3/2)p(x_1), \qquad p(x_3) \leqq 2p(x_2) \leqq 3p(x_1).$$

Now

$$\begin{aligned}p(x_3)^{-1}p'(x_3) &= p(x_3)p(x_3)^{-2}p'(x_3) \\ &\leqq Kp(x_3)p(x_1)^{-2}p'(x_1) \\ &\leqq 3Kp(x_1)^{-1}p'(x_1).\end{aligned}$$

Thus (H_4) holds, and by Corollary 2 the conclusion follows.

In particular, (H_5) holds if $p'(x)/p(x)^2$ is nonincreasing for all x above some $\bar{x}$.

L. A. Gusarov has proved [**4**] that all solutions of (1) tend to zero as $x \to \infty$ under a set of hypotheses including (in our notation) (H_1) and the hypothesis that pp' is of bounded variation on some half-line $[\bar{x}, \infty)$. Under these conditions $p(x)p'(x)$ has a finite limit as $x \to \infty$, and this limit is nonnegative. Corollary 4 includes all those examples in which the limit of pp' is positive, and some (but not all) of the examples in which the limit is 0.

EXAMPLES. If $p(x) = x^a (a > 0)$, p'/p^2 is nonincreasing, and by the remark after Corollary 3, (H_5) is satisfied. If we define $\exp_n$ inductively by

$$\exp_1 x = \exp x, \ \exp_{n+1} x = \exp(\exp_n x),$$

and define $\log_n$ analogously, the functions $p(x) = \exp_n x$, $p(x) = \log_n(x)$ (n a positive integer) have p'/p^2 decreasing above a certain $\bar{x}$, and again (H_5) holds.

If p is continuous and nondecreasing and coincides with $\exp x$ at $x_n = \log(n/2)$ ($n = 2, 3, 4, \cdots$), we have $p(x) < 2p(x_n)$ on $[x_n, x_{n+1}]$, so

$$\int_{x_n}^{x_{n+1}} p(x)\,dx < 2\int_{x_n}^{x_{n+1}} \exp x\,dx = 1.$$

Thus if sequence (14) satisfies (15) with $\delta = 1/10$, each interval $[b_j, c_j]$ contains one of the intervals $[x_n, x_{n+1}]$; each term in the sum in (5) is at least 1/2, and (H_3) holds. On the other hand, if we choose

$$b_j = x_j, \qquad c_j = x_j + (1 - 1/j)(x_{j+1} - x_j),$$

we can choose $p(x)$ constant on each interval $[b_j, c_j]$. This sequence satisfies (16) with every positive δ; in fact,

$$\lim_{n\to\infty} \sum_{j=1}^{n} (b_{j+1} - c_j)/b_{n+1} = 0.$$

But (5) is not valid, so the special case mentioned in the paragraph after Corollary 1 does not apply. Neither does Corollary 2 nor Corollary 3.

Added in proof. My attention has been called to a discussion of this problem in L. Cesari, *Asymptotic behavior and stability problems in ordinary differential equations*, Ergebnisse der Mathematik und ihre Grenzgebiete N.F., Heft 16, Springer Verlag, Berlin, 1959, pp. 80

et seq. In particular Cesari states that L. Tonelli and G. Sansone independently (in a publication unavailable to me) established Armellini's result (cf. remark after Corollary 1), and Sansone also established another condition ensuring $y(x)\to 0$ which does not seem to follow from our theorem.

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