REFERENCES

1. G. Polya, *On the mean-value theorem corresponding to a given linear homogeneous differential equation*, Trans. Amer. Math. Soc. **24** (1922), 312–324.

2. E. F. Beckenbach and R. Bellman, *Inequalities*, Springer, Berlin, 1961.

3. F. R. Gantmacher and M. G. Krein, *Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme*, Akademie-Verlag, Berlin, 1960.

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A VOLTERRA EQUATION WITH A VERY SINGULAR KERNEL[1]

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For continuous, or mildly singular kernels, the Volterra equations of second kind can be solved with a Neumann series which converges for the entire complex λ plane. The purpose of the present note is to show, by studying a special case, what can happen if the kernel is very singular. In particular, we find that the equation can have finite spectral values, in fact the circle $|\lambda| = 1$ can be a natural boundary for the resolvent kernel, and that it is not easy to describe a "natural" domain and range for the operator in terms of the usual classes of functions.

Consider the Volterra type integral equation for complex valued functions of a real variable given by

$$\phi = a + \lambda k * \phi, \tag{1}$$

where

$$k(t) = (i/\pi^{1/2})t^{-3/2} \exp(i\pi/4 + i/t) \tag{2}$$

and $*$ denotes the convolution,

$$f * g = \int_0^t f(t - s)g(s)\, ds.$$

(The study of the boundary value problem for the equation of vibration of elastic bars leads to the related equation

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$$\phi = a + \lambda k * \bar{\phi}$$

which is solved in [1] and which does not exhibit the pathologies discussed here.)

By formal substitution we obtain the Neumann series

$$\phi = a + \lambda k * a + \lambda^2 k * (k * a) + \cdots . \tag{3}$$

It has been proved in [1, Appendix I] that the integral $k * a$ exists for $0 \leqq t \leqq T$ if a is CBV[0, T],[2] that for this class of functions

$$k * (k * a) = (k * k) * a \equiv k_2 * a,$$

and that

$$k_2(t) = (2i/\pi^{1/2})t^{-3/2} \exp(i\pi/4 + 4i/t).$$

Since $k_2(t)$ has the same order singularity as $k(t)$, the above results assure that $k_2 * a$ exists and

$$k * (k * (k * a)) = (k * k_2) * a \equiv k_3 * a.$$

k_3 can also be evaluated as above; in fact, by [1, Appendix I], (3) can be written

$$\phi = a + \sum_{n=1}^{\infty} \lambda^n k_n * a \tag{4}$$

where

$$k_n(t) = (ni/\pi^{1/2})t^{-3/2} \exp(i\pi/4 + in^2/t). \tag{5}$$

We prove

THEOREM 1. *If $a(t)$ is CBV[0, T] and $|\lambda| < 1$, then* (4) *defines a continuous solution of* (1).

By integration by parts

$$\begin{aligned} \int_0^t k_n(t-s)a(s)\,ds = n \exp(i\pi/4)\pi^{1/2} \Big\{ &(i/n^2) \exp(in^2/t)t^{1/2}a(0) \\ &+ (i/n^2) \int_0^t \exp(in^2/(t-s))[-1/2(t-s)^{-1/2}a(s)\,ds + (t-s)^{1/2}\,da(s)] \Big\}. \end{aligned} \tag{6}$$

Thus with $A = \max_{[0,T]} a(t)$

$$\begin{aligned} |k_n * a| \leqq\ & (1/(n\pi^{1/2})) \\ & \cdot \left\{ AT^{1/2} + A \int_0^t \frac{1}{2}(t-s)^{-1/2}\,ds + \int_0^t (t-s)^{1/2}\,dV(a;0,s) \right\}, \end{aligned}$$

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[2] CBV[0, T] means that class of functions which are continuous and of bounded variation on [0, T].

where $V(a; 0, s)$ is the total variation of a from 0 to s. Hence

$$|k_n * a| \leqq (1/n)(2A + V(a; 0, T))(T/\pi)^{1/2} = B/n$$

where B is independent of t and n. Therefore

$$\left|\sum_{n=1}^{\infty} \lambda^n k_n * a\right| \leqq B \sum_{n=1}^{\infty} |\lambda|^n/n$$

so that (4) converges uniformly and absolutely on $[0, T]$.

We must yet show that ϕ defined by (4) is continuous and that (1) is satisfied. Suppose $h > 0$; we have

$$\begin{aligned} D = & \int_0^{t+h} s^{-3/2} \exp(in^2/s) a(t + h - s)\, ds \\ & - \int_0^t s^{-3/2} \exp(in^2/s) a(t - s)\, ds \\ = & \int_0^t s^{-3/2} \exp(in^2/s)[a(t + h - s) - a(t - s)]\, ds \\ & + \int_t^{t+h} s^{-3/2} \exp(in^2/s) a(t + h - s)\, ds. \end{aligned}$$

By a mean value theorem for BV functions [2, p. 623],

$$\begin{aligned} |D| \leqq & \{|a(t + h) - a(t)| + V(a(s); t, t + h) + V(a(s); 0, h)\} M_1 \\ & + \{|a(h)| + V(a(s); 0, h)\} M_2 \end{aligned}$$

where

$$M_1 = \sup_{0 \leqq t_1 < t_2 \leqq T} \left| \int_{t_1}^{t_2} s^{-3/2} \exp(in^2/s)\, ds \right|$$

and

$$M_2 = \sup_{t \leqq t_1 < t_2 \leqq t+h} \left| \int_{t_1}^{t_2} s^{-3/2} \exp(in^2/s)\, ds \right|.$$

Now, a and $V(a)$ are continuous on $[0, T]$ and, since the integral exists, $M_2 = o(1)$. Thus $|D| \to 0$ as $h \to 0$. For $h < 0$, we have the corresponding result so that each term in (4) is continuous. This implies, with the uniform convergence, that ϕ is continuous.

Also,

$$\lambda k * \phi = \lambda k * a + \lambda k * (\sum \lambda^n k_n * a)$$
$$= \sum \lambda^n k_n * a = -a + \phi$$

so that (4) is a solution of (1).

Using (6), it is not difficult to show that, for $|\lambda| < 1$, (4) can be written

$$\phi = a + \int_0^t R(\lambda; t-s) a(s)\, ds, \tag{7}$$

where

$$R(\lambda; t) = (1/\pi^{1/2}) e^{i\pi/4} t^{-3/2} \sum_{n=1}^{\infty} n\lambda^n \exp(in^2/t).$$

By [3, Theorem 8, p. 424], *the circle* $|\lambda| = 1$ *is a natural boundary of* $R(\lambda, t)$.

To further illustrate the behavior of this equation, we give an elementary proof that $\lambda = 1$ is a spectral value of the equation. We assume that $\lambda = 1$ and $a \equiv 1$ so that (4) gives

$$\phi(t) = 1 + \sum_{n=1}^{\infty} \int_0^t k_n(s)\, ds. \tag{8}$$

We let $u = s^{-1}$ and integrate by parts twice to obtain

$$\int_0^t k_n(s)\, ds = C t^{1/2} n^{-1} \exp(in^2/t) + O(n^{-3})$$

uniformly in n. Set $t = 1/2\pi m$ where m is any positive integer. Then

$$\int_0^t k_n(s)\, ds = C \frac{1}{(2\pi m)^{1/2}} \frac{1}{n} + O(n^{-3}),$$

and

$$\phi\left(\frac{1}{2\pi m}\right) = 1 + C' \sum n^{-1} + \sum O(n^{-3}).$$

Since the second series converges and the first diverges, we see that the series (8) fails to define $\phi(t)$ for $t = 1/2\pi m$.

On the other hand, we can show that: *if* $\lambda = e^{i\theta}$ *and* $a(t)$ *is differentiable with* $a' \in CBV[0, T]$ *and* $a(0) = 0$, *then* (4) *defines a continuous solution of* (1).

With the above hypothesis, (6) can be written

$$k_n * a = Cn^{-3}\left\{\exp\left(\frac{in^2}{t}\right)\left[-\frac{1}{2}t^{3/2}a(t) + t^{5/2}a'(t)\right] + \int_0^t \exp\left(\frac{in^2}{t-s}\right)\left[\frac{3}{4}(t-s)^{1/2}a(s)\,ds - 3(t-s)^{3/2}a'(s)\,ds + (t-s)^{5/2}\,da'(s)\right]\right\}.$$

As in the proof of Theorem 1, we obtain $|k_n * a| \leqq Bn^{-3}$ where B is independent of n and t. Thus the series (4) converges uniformly and absolutely so that the conclusions follow exactly as in Theorem 1.

References

1. E. L. Roetman, *Vibration of elastic bars*, Ph.D. Thesis, Oregon State University, Corvallis, Ore., 1963.

2. E. W. Hobson, *Theory of functions of a real variable*, Vol. 1, Dover, New York, 1957.

3. R. Cooper, *The behavior of certain series associated with limiting cases of elliptic theta-functions*, Proc. London Math. Soc. (2) **27** (1928), 410–426.

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