INCLUSION RELATIONS AMONG ORLICZ SPACES

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This paper contains two results; the first extends to a wide class of Orlicz spaces the statement, due to Krasnosel'skii and Rutickii [1, p. 60], that $L_1$ is the union of the Orlicz spaces which it contains properly; the second shows that for a wide class of spaces this is not true, i.e. there exists a set of Orlicz spaces no one of which is the union of the Orlicz spaces it contains properly. Here the Orlicz spaces are defined on $[0, 1]$ which is given Lebesgue measure $\mu$.

1. We give in this section several definitions together with some elementary results about Orlicz spaces and convex functions.

Let $C$ be the set of convex symmetric functions $\Phi : (—\infty, \infty) \to [0, \infty)$ such that $\Phi(0) = 0$, $\lim_{s \to 0} \Phi(s)/s = 0$ and $\lim_{s \to \infty} \Phi(s) = \infty$. If $\Phi$ and $\Omega$ are two elements of $C$, we say $\Phi \leq \Omega$ if there exist constants $c$ and $s_0$ such that $\Phi(s) \leq \Omega(cs)$ for all $s \geq s_0$. We say $\Phi \sim \Omega$ if $\Phi \leq \Omega$ and $\Omega \leq \Phi$; we say $\Phi < \Omega$ if $\Phi \leq \Omega$ but $\Omega \not\leq \Phi$. If $\Phi_1 \sim \Phi_2$ and $\Phi_1 \leq \Omega_1$ ($\Phi_1 < \Omega_1$) then $\Phi_2 \leq \Omega_2$ ($\Phi_2 < \Omega_2$).

If $\Phi \in C$, then there exists a nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(0) = 0$, $\lim_{s \to \infty} \phi(s) = \infty$ and

$$\Phi(s) = \int_0^{|s|} \phi(t) \, dt$$

(see [1, p. 5]). This representation for $\Phi$ yields easily the two following inequalities:

(1) $$\frac{s}{2} \phi\left(\frac{s}{2}\right) \leq \Phi(s) \leq \phi(s),$$

(2) $$2\Phi(s) \leq \Phi(2s).$$

Let

$$L^\Phi_\bullet = \{f \in L_1 : \Phi(cf) \in L_1 \text{ for some positive real number } c\}.$$

The set $L^\Phi_\bullet$ is called an Orlicz space. It has a unique uniformity which is compatible with the order relation. Since this uniformity does not intervene in what follows, we do not give its definition; for this see [1, p. 69].

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The order relations among the elements of $\mathcal{C}$ give rise to order relations among the $L^*_\Phi$ as follows:

(a) $\Phi \leq \Omega$ implies $L^*_\Omega \subset L^*_\Phi$.

(b) $\Phi \sim \Omega$ implies $L^*_\Phi = L^*_\Omega$.

(c) $\Phi < \Omega$ implies $L^*_\Omega \subset L^*_\Phi$ but $L^*_\Phi \not\subset L^*_\Omega$.

Statements $a$ and $b$ are direct consequences of the definitions; while $c$ is a special case of

(d) $\limsup_{s \to \infty} \frac{\Omega(\alpha s)}{\Phi(s)} = \infty$ for all $\alpha > 0$ implies there exists $f \in L^*_\Phi$ such that $f \notin L^*_\Omega$.

**Proof.** Let $E_{ij}$ be a pairwise disjoint double sequence of intervals in $[0,1]$ such that $\mu(E_{ij}) \neq 0$, $i, j = 1, 2, \cdots$. For each pair of natural numbers $(n, i)$ there exists a number $s_{ni} > 0$ such that $s > s_{ni}$ implies $\Phi(s)\mu(E_{ni}) > 2^{-i}n^{-2}$. There exist numbers $c_{ni} > s_{ni}$ such that $\Phi(c_{ni})/n > n^2\Phi(c_{ni})$.

Let $E_{ni}$ be a nonempty subinterval of $E_{ni}$ such that $\Phi(c_{ni})\mu(E_{ni}) = 2^{-i}n^{-2}$ and define

$$f(x) = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} c_{ij} X_{E_{ij}}(x).$$

It is easy to show that $f \in L^*_\Phi$ but $f \notin L^*_\Omega$.

(e) One can use $a$, $b$ and $d$ to show that $\Phi < \Omega$ if $L^*_\Omega$ is a proper subset of $L^*_\Phi$.

2. We say that $\Phi \in \mathcal{C}$ satisfies $\ast$ if

$$\ast \quad \limsup_{s \to \infty} \frac{\Phi(2s)}{\Phi(s)} < \infty$$

and we say it satisfies $\ast\ast$ if

$$\ast\ast \quad \liminf_{s \to \infty} \frac{\Phi(2s)}{s\Phi(s)} > 0.$$

These conditions are similar to the $\Delta_2$ and $\Delta_3$ conditioned in [1]. A function $\Phi$ which satisfies $\ast$ grows less rapidly than some power and in addition it grows regularly; while a function $\Phi$ which satisfies $\ast\ast$ grows like exp. It follows that $\mathcal{C}$ contains functions which satisfy neither $\ast$ nor $\ast\ast$.

**Theorem 1.** Suppose $\Phi \in \mathcal{C}$ and $\Phi$ satisfies $\ast$; then, $L^*_\Phi$ is the union of the Orlicz spaces it contains properly.

**Proof.** Let $f \in L^*_\Phi$. We will prove there exists $\Omega$ in $\mathcal{C}$ such that $\Phi < \Omega$ and such that $f \in L^*_\Omega$. If $f \in L_{\infty}$, we are finished because $L_{\infty} \subset L^*_\Phi$ properly. We will assume that $f(x) \geq 0$ a.e.; this is in order because
$f \in L_\Phi^*$ if $|f| \in L_\Phi^*$. Let $c$ be a positive real number such that $\Phi(cf) \in L_1$. Define $\nu(s) = \mu((x: \Phi(cf(x)) \leq s))$ and recall that

$$
\int_0^1 u(\Phi(cf)) \, d\mu = \int_0^\infty u(s) \, d\nu(s)
$$

whenever $u$ is integrable with respect to $d\nu$. The $sd\nu(s)$ measure of $[0, \infty)$ is finite (let $u(s) = s$) so there exists a function $\omega: [0, \infty) \to [0, \infty)$ such that

(a') $\int_0^\infty s\omega(s) \, d\nu(s) < \infty$,
(b') $\omega$ is nondecreasing,
(c') $\omega(0) = 0$ and $\lim_{s \to \infty} \omega(s) = \infty$.

The function

$$
\Omega_0(s) = \int_0^{|s|} \omega(|s|) \, ds
$$

is an element of $\mathcal{C}$ and so is $\Omega(s) = \Omega_0(\Phi(s))$ [1, p. 10]. The inequality 1 of §1 gives

$$
\int_0^1 \Omega(cf) \, d\mu = \int_0^\infty \Omega(s) \, d\nu(s) \leq \int_0^\infty s\omega(s) \, ds < \infty
$$

from which it follows that $f \in L_\Omega^*$. To complete the proof, we must show $\Omega > \Phi$. Using inequality 1 again we get

(3) $\Omega(s) = \Omega_0(\Phi(s)) \geq [\Phi(s)/2] \omega(\Phi(s)/2)$;

together with 2 this gives $\Omega(2s) \geq \Phi(s)$ whenever $s \geq s_0$. Here $s_0$ is any positive number such that $\omega(\Phi(s_0)/2) \geq 1$. This shows that $\Omega \geq \Phi$.

Let $\alpha$ be any positive number. There exist positive numbers $s_0$ and $M(\alpha)$ such that $\Phi(\alpha s) \geq M(\alpha) \Phi(s)$ if $s \geq s_0$; this is true because $\Phi$ satisfies *. Now this with 3 gives $\Omega(\alpha s) \geq M(\alpha) \Phi(s) \omega(\Phi(\alpha s)/2) / 2$ for $s \geq s_0$ and this in turn gives

$$
\limsup_{s \to \infty} \frac{\Omega(\alpha s)}{\Phi(s)} = \limsup_{s \to \infty} \frac{M(\alpha) \omega(\Phi(\alpha s)/2)}{2} = \infty.
$$

Because $\alpha$ was arbitrary, we have that $\Omega \not\geq \Phi$.

**Lemma.** Suppose $\Omega$ and $\Phi$ are two elements of $\mathcal{C}$ such that for some $\alpha > 1$

$$
\limsup_{s \to \infty} \frac{\Omega(s)}{\Phi(\alpha s)} \geq 1.
$$

Then, if $t_0$ is any positive number there exists $t \geq t_0$ such that $\Omega(s) \geq \Phi(s)$ for all $s \in [t, \alpha t]$. 

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Proof. Let $t_0$ be any positive number and let $t \geq t_0$ be any number such that $\Omega(t) \geq \Phi(at)$. Let $l'$ be a straight line through $(t, \Omega(t))$ which lies beneath the graph of $\Omega$; such a line exists because $\Omega$ is convex ($l'$ is not necessarily unique). Let $l$ be the straight line, parallel to $l'$ which passes through $(t, \Phi(at))$. Let $u, v$ be the two numbers such that $u < v$ and $l$ passes through the points $(u, \Phi(u)), (v, \Phi(v))$. By comparing similar triangles we get

$$\frac{\Phi(v) - \Phi(u)}{v - u} = \frac{\Phi(at) - \Phi(u)}{t - u}.$$ 

This leads directly to the inequality $v > at$. For $s \in [t, at]$, $(s, \Phi(s))$ is beneath the line $l$ while $(s, \Omega(s))$ is above the line $l'$. Hence $\Phi(s) \leq \Omega(s)$ for $s \in [t, at]$.

Theorem 2. Suppose $\Phi \in C$ and satisfies $\ast\ast$; then $L^*_\Phi$ is not the union of the Orlicz spaces it contains properly.

Proof. The condition $\ast\ast$ implies there exists $\alpha > 0$ and $s_0$ such that $s \geq s_0$ implies $\Phi(2s) > s\Phi(s)$. Let $c_n = 2^n s_0$. Let $(E_n)$ be a sequence of pairwise disjoint subintervals of $[0, 1]$ such that $\Phi(c_n)\mu(E_n) = 2^{-n}$. This is possible because

$$\Phi(c_n) \geq 2^{n(n-1)/2} s_0^{-n} \alpha \Phi(s_0).$$

If we set

$$f(x) = \sum_{n=1}^{\infty} c_n X_{E_n}(x),$$

then $\Phi(f) \in L_1$ while $\Phi(2f) \notin L_1$; in fact

$$\int_{E_n} \Phi(2f) \, d\mu \geq 2^n s_0 \alpha \Phi(c_n)\mu(E_n) = a s_0.$$

Suppose $L^*_\Omega$ is a proper subset of $L^*_\Phi$. This is the case only if $\Phi \subset \Omega$. Let $k$ be any number in $(0, 1)$; we will show that $\Omega(kf) \in L_1$ and hence that $f \notin L_\Omega$. The assertion $\Omega > \Phi$ implies that

$$\limsup_{s \to \infty} \frac{\Omega(ks)}{\Phi(2s)} = \infty.$$ 

Let $t_0 = c_1$ and apply the lemma to find $t_1$ such that $s \in [t_1, 2t_1]$ implies $\Omega(ks) \geq \Phi(2s)$. Having chosen $t_n$, choose $t_n > 2t_{n-1}$ such that $s \in [t_n, t_{2n}]$ implies $\Omega(ks) \geq \Phi(2s)$. By induction this yields an infinite sequence of intervals $[t_n, 2t_n]$ each of which contains one of the numbers $c_m$. 

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The proof is completed by observing that
\[ \int_0^1 \Omega(kf) \, df \geq \sum \Phi(2c_m) \mu(E_m) \geq \sum_{n=1}^{\infty} \alpha s_0. \]

Reference

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**ON A COMBINATORIAL PROBLEM OF ERDÖS**

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Let \( C(n, m) \) denote the binomial coefficient \( n!/(m!n-m!) \). Let \( S \) be a set containing \( N \) elements and let \( X \) be a collection of subsets of \( S \) with the property that if \( A, B \) and \( C \) are distinct elements of \( X \), then \( A \cup B \neq C \). Erdős [1], [2], has conjectured that \( X \) contains at most \( KC(N, [N/2]) \) elements where \( K \) is a constant independent of \( X \) and \( N \). The problem is related to a result of Sperner [3] to the effect that if the collection \( X \) has the more restrictive property that no element of \( X \) contains any other, then \( X \) can have at most \( C(N, [N/2]) \) elements.

We show below that Erdős’ conjecture for \( K = 2^{3/2} \) can be deduced directly from Sperner’s result.

Let \( L_N \) be defined by
\[ L_N = 2^{1N/2} C(N - [N/2], [N/2]) + 2^{N-[N/2]} C([N/2], [N/4]). \]

An easy calculation shows that \( L_N \) is always less than \( 2^{3/2} C(N, [N/2]) \) to which it is asymptotic for large \( N \). We prove:

**Theorem.** If \( X \) is a family of subsets of an \( N \) element set \( S \) such that no three distinct \( A, B, C \) in \( X \) satisfy \( A \cup B = C \), then \( X \) has less than \( L_N \) elements.

**Proof.** For any finite set \( T \) and family \( X \) of subsets of \( T \) define
\[ m_T(X) \equiv \{ A \in X \mid B \in X \text{ and } B \subseteq A \text{ imply } B = A \}. \]

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