A NOTE ON THE RECURSIVE UNSOLVABILITY OF
PRIMITIVE RECURSIVE ARITHMETIC

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We wish to show the recursive unsolvability of primitive recursive arithmetic (PRA). By PRA we mean a quantifier-free formal system of arithmetic which has expressions for all primitive recursive functions. In such a system all valid variable free formulas are provable and both of the Gödel incompleteness theorems hold. Further, we may define in the system bounded quantifiers and (for a suitable Gödel numbering) the following primitive recursive functions: \( \text{th}(x) \), a function which enumerates the Gödel numbers of theorems of PRA, and \( \text{sub}(n, m) \), the function whose value is the Gödel number of the formula obtained by replacing the first variable in alphabetic order by the numeral \( n \) through the formula number \( m \).

If there is a recursive decision procedure for PRA, then the set of Gödel numbers of nontheorems is recursively enumerable. But if a set is recursively enumerable then it is primitive recursively enumerable. Thus if PRA is solvable there is a primitive recursive function whose range is precisely the set of Gödel numbers of nontheorems.

Assume there exists such a function \( f \). Consider the formula

\[
\text{th}(x) = \text{sub}(x_0, x_0) \supset (Ez). \ z \leq x \land f(z) = \text{sub}(x_0, x_0).
\]

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Let the Gödel number of this formula be \( i \), then the formula whose Gödel number is \( \text{sub}(i, i) \) will be

\[
(2) \quad \text{th}(x) = \text{sub}(i, i) \supset (\exists z). \ z \leq x \land f(z) = \text{sub}(i, i).
\]

Suppose this formula is provable. There must be some \( k \) such that \( \text{th}(k) = \text{sub}(i, i) \) is valid and hence provable. But if this formula and (2) are provable then by modus ponens and substitution \((\exists z). \ z \leq k \land f(z) = \text{sub}(i, i)\). Thus \( \text{sub}(i, i) \) is one of the first \( k \) nontheorems, contrary to hypothesis.

Suppose (2) is not a theorem. By hypothesis, \( f \) enumerates all nontheorems, so there must be some \( n \) such that \( f(n) = \text{sub}(i, i) \). But now we may obtain a proof of (2) as follows: For each number \( m \) less than \( n \), \( \text{th}(m) \neq \text{sub}(i, i) \) will be provable, thus the conjunction of these \( n \) formulas will be provable. But this gives \( \text{th}(x) \neq \text{sub}(i, i) \lor n \leq x \), and hence \( \text{th}(x) \neq \text{sub}(i, i) \lor n \leq x \land f(n) = \text{sub}(i, i) \), from which (2) follows immediately.

Thus the formula (2) is neither provable nor unprovable, so there must be no such formula and PRA is unsolvable.